Tel-Aviv University – School of Mathematical Science

On Shift Invariant Spaces, Wavelets and Subdivision in the absence of Two-Scale Refinability

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by

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This thesis was carried out under the supervision of Prof. Dany Leviatan.

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Preface

The motivation for this work is a recently constructed ([BTU]) family of generators of shift-invariant spaces with certain optimal approximation properties, but which are not refinable in the classical sense. We try to see whether, once the classical refinability requirement is removed, it is still possible to construct meaningful wavelets and subdivision schemes that are well suited for applications. In the introduction we lay out the background and the motivation for this research. In Chapter 2 we present the basic theory of the structure of shift invariant spaces which serves as framework throughout the work. We also present some new "regularity" results that are required for the wavelet constructions of Chapter 5. In Chapters 3 and 4 we discuss new generalizations of two-scale refinability and subdivision to multi-scale refinability and subdivision. In Chapter 5 we construct non-stationary wavelet decompositions of shift invariant spaces which are not constraint to be two-scale refinable. In Chapter 6 we first present the basic theory of approximation from shift invariant spaces and a few new results in this field. We then proceed to justify the decompositions of Chapter 5, by showing that the constructed non-stationary wavelets inherit the good approximation properties of the decomposed non-refinable shift invariant space.

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1 Introduction

In classical refinable wavelet theory ([Ch], [Da], [Ma]) one begins with a **finitely generated shift invariant** (FSI) space $S(\Phi) := \overline{span}\{\phi(\cdot - k) | \phi \in \Phi, k \in \mathbb{Z}^d\}$, where Φ is a finite set and the closure is taken in some Banach space X. Typically, $S(\Phi)$ is selected to have **approximation order** $m \in \mathbb{N}$. This means that for any $h > 0$ and $f \in X$

$$E\left(f, S(\Phi)^h\right)_X := \inf_{g \in S(\Phi)^h} \|f - g\|_X \le Ch^m |f|_X, \tag{1.1}$$

where

$$S(\Phi)^h := \overline{span}\left\{\phi\left(h^{-1} \cdot - k\right) | \phi \in \Phi, k \in \mathbb{Z}^d\right\},$$

and $|\cdot|_X$ is a semi-norm, measuring the smoothness of the elements of X.

To allow the construction of wavelets associated with $S(\Phi)$, one assumes that the shift invariant space is **two-scale refinable**, namely

$$S(\Phi) \subset S(\Phi)^{1/2}. \tag{1.2}$$

One then selects a complementary set of generators, so called **wavelets**, Ψ so that

$$S(\Phi)^{1/2} = S(\Phi) + S(\Psi). \tag{1.3}$$

It is easy to see that (1.3) can be dilated to any given scale $J \in \mathbb{Z}$ such that

$$S(\Phi)^{2^{-J}} = S(\Phi)^{2^{-J+1}} + S(\Psi)^{2^{-J+1}}.$$

Assume $f_\Phi^J \in S(\Phi)^{2^{-J}}$ such that $f_\Phi^J = f_\Phi^{J-1} + f_\Psi^{J-1}$ where $f_\Phi^{J-1} \in S(\Phi)^{2^{-J+1}}$, $f_\Psi^{J-1} \in S(\Psi)^{2^{-J+1}}$. Then, f_Φ^{J-1} plays the role of a low resolution approximation to f_Φ^J, while f_Ψ^{J-1} is the difference between the two. Typically, if f_Φ^J is a sufficiently smooth function or J is sufficiently large, then $f_\Phi^{J-1} \approx f_\Phi^J$ and $f_\Psi^{J-1} \approx 0$. Under certain conditions (1.3) leads to a wavelet decomposition

$$S(\Phi)^{2^{-J}} = S(\Psi)^{2^{-J+1}} + S(\Psi)^{2^{-J+2}} + S(\Psi)^{2^{-J+3}} + \cdots, \tag{1.4}$$

such that for any $f_\Phi^J \in S(\Phi)^{2^{-J}}$ there exists a decomposition

$$f_\Phi^J = f_\Psi^{J-1} + f_\Psi^{J-2} + f_\Psi^{J-3} + \cdots. \tag{1.5}$$

In applications FSI spaces are used as follows: Let f be some signal that one wishes to approximate. Using property (1.1), one chooses a fine enough scale $J \in \mathbb{Z}$ and computes an approximation

$$f \approx f_{\Phi}^{J} \in S(\Phi)^{2^{-J}}. \tag{1.6}$$

In some applications there is no need to further decompose the approximation f_{Φ}^{J} to the wavelet sum (1.5). Typical examples are curve and surface (linear) approximations in CAGD or re-sampling in image processing. The wavelet decomposition (1.4) is effective in applications that require a compact representation of the signal such as compression, denoising, segmentation, etc.

Let $S(\Phi_0)$ be a non-refinable FSI space. Namely, $S(\Phi_0) \not\subset S(\Phi_0)^{1/2}$. There are many examples of non-refinable FSI spaces that perform well in approximations of type (1.6). In fact, there is an interesting recent construction ([BTU]) of shift invariant spaces that are "optimal" in some approximation theoretical sense and are not two-scale refinable. Nevertheless, we would still like to decompose the space $S(\Phi_0)^{2^{-J}}$ into a sum of difference (wavelet) spaces in the sense of (1.4) (see [CSW] for a different approach). Since our FSI space is not refinable we need to replace $S(\Phi_0)$ by a different space $S(\Phi_1)$ to play the role of a low resolution and a (wavelet) space $S(\Psi_1)$ to serve as a difference space in a decomposition similar to (1.3)

$$S(\Phi_0)^{1/2} = S(\Phi_1) + S(\Psi_1).$$

In this work we show that such meaningful decomposition techniques exist. They allow us, to further decompose $S(\Phi_1)^{1/2} = S(\Phi_2) + S(\Psi_2)$ and so on and obtain a non-stationary wavelet decomposition similar to (1.4)

$$S(\Phi_0)^{2^{-J}} = S(\Psi_1)^{2^{-J+1}} + S(\Psi_2)^{2^{-J+2}} + S(\Psi_3)^{2^{-J+3}} + \cdots$$

Thus, the (non-stationary) sequence $\{\Phi_j\}$ is a means to obtain the non-stationary wavelet sequence $\{\Psi_j\}$. The sequence $\{\Phi_j\}$ is also used to determine the (linear) approximation properties of the wavelets. It is interesting to note that our techniques are able to recover the stationary choice $\Phi_j = \Phi_0$, $\Psi_j = \Psi$, whenever $S(\Phi_0)$ is two-scale refinable and $S(\Phi_0)^{1/2} = S(\Phi_0) + S(\Psi)$.

Another interesting question addressed in this work is the following: Let $S(\Phi_0)$ be an "optimal" non-refinable FSI space under some approximation theoretical gauge. Obviously, if $S(\Phi_0)$ has an "optimal" approximation property, any constructed $S(\Phi_1) \subset S(\Phi_0)^{1/2}$ cannot inherit this exact "optimal" property. One then asks how close are the approximation properties of $S(\Phi_1)$ to those of $S(\Phi_0)$? Another question is: In what way (if any) are wavelets that decompose dilations of "optimal" non-refinable FSI spaces better than known existing wavelets?

Finally, the existence of non-refinable FSI spaces with good approximation properties leads to the following question. Is it possible, by relaxing the constraint of classical two-scale refinability, to generalize the notion of two-scale subdivision schemes ([CDM], [Dy]), such that we can construct new non-refinable generators of FSI spaces with interesting properties?

In this work we relax the classical two-scale refinability requirement (1.2) (see also [Der]) to the following form of **multi-scale refinability**

$$S(\Phi) \subset S(\Phi)^{2^{-1}} + S(\Phi)^{2^{-2}} + \cdots + S(\Phi)^{2^{-(M-1)}}, \qquad 2 \le M \in \mathbb{N}. \qquad (1.7)$$

For example, if $S(\phi)$ is a Principal Shift Invariant (PSI) space, a shift invariant space generated by one function, then (1.7) is equivalent to the existence of a **multi-scale relation**

$$\phi = \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} p_{m,k} \phi(2^m \cdot - k). \qquad (1.8)$$

We shall see that (1.8) indeed leads, in a natural way, to the notion of **multi-scale subdivision**.

A classical two-scale subdivision scheme is a process, where the next level of refinement is computed from the current level using a single given mask $P = \{p_k\}_{k \in \mathbb{Z}^d}$. With each such convergent scheme there is an associated refinable function $\phi \in C(\mathbb{R}^d)$ for which a two-scale relation holds

$$\phi = \sum_{k \in \mathbb{Z}^d} p_k \phi(2 \cdot - k).$$

The generalized (multi) M – scale subdivision scheme computes the next level of refinement from the previous $M-1$ levels, using $M-1$ given masks, $P_m = \{p_{m,k}\}_{k \in \mathbb{Z}^d}$, $m = 1, \ldots, M-1$. We show that each such convergent scheme is associated with a function $\phi \in C(\mathbb{R}^d)$ for which an M – scale relation (1.8) holds.

2 Shift invariant spaces

Shift invariant spaces are a special case of invariant subspaces in Banach spaces. Here we use the framework of [BDR2] and present results that are required for the constructions in Chapter 5. We follow the notation in [BDR2] with the convention that some of the equalities hold up to a set of measure zero.

Definition 2.1 For any $k \in \mathbb{Z}^d$ we denote the linear **shift operator** S_k by

$$S_k(f) := f(\cdot - k).$$

Definition 2.2 Let V be a closed subspace of $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$. We say that V is a **shift invariant** (SI) space if it is invariant under the operators $\{S_k \mid k \in \mathbb{Z}^d\}$. We say that a set Φ **generates** V if $V = S(\Phi) := \overline{span}\{\phi(\cdot - k) \mid \phi \in \Phi, k \in \mathbb{Z}^d\}$. We say that V is a **finite shift invariant** (FSI) space, if there exists a finite generating set $\Phi = \{\phi_i\}_{i=1}^n$, such that $V = S(\Phi)$. In such a case we say that V is of **length** $\le n$. We denote $len(V) := \min\{|\Phi| \mid V = S(\Phi)\}$. V is called a **principal shift invariant** (PSI) space if $len(V) = 1$.

To approximate functions with arbitrary precision one uses dilates of shift invariant spaces.

Definition 2.3 Let V be an SI space and $h > 0$. We denote by V^h the dilated closed space

$$V^h := \{\phi(\cdot / h) \mid \phi \in V\}.$$

The following very useful result shows that orthogonal projection into SI spaces and shift operators commute in $L_2(\mathbb{R}^d)$. We state and prove a slightly more general result.

Theorem 2.4 Let H be an Hilbert space and $V \subseteq H$ an invariant closed subspace of A, A^*, where $A: H \to H$ is a bounded linear operator. Then for any $f \in H$

$$P_V \circ A(f) \underset{H}{=} A \circ P_V(f), \tag{2.1}$$

where P_V is the orthogonal projection onto V.

Proof If $V = H$ then we are done. Else $V^{\perp}$ is not empty. Since V is invariant under A, for any $g \in V$ we also have $Ag \in V$ and so $P_V \circ A(g) = A \circ P_V(g) = Ag$. Thus (2.1) is correct for any element in V. It is obvious that for any $0 \neq g \in V^{\perp}$ we have $A \circ P_V(g) = A0 = 0$. We claim that $P_V \circ A(g) = 0$ also. Otherwise there exists $0 \neq \phi \in V$ such that $\langle Ag, \phi \rangle \neq 0$. But this implies that for the element $A^* \phi \in V$ we have $\langle g, A^* \phi \rangle = \langle Ag, \phi \rangle \neq 0$, which contradicts the assumption $0 \neq g \in V^{\perp}$. Thus (2.1) is also correct for any function in $V^{\perp}$. We can now proceed with the obvious decomposition technique: any $f \in H$ has a (unique) representation $f = f_1 + f_2$ with $f_1 \in V$, $f_2 \in V^{\perp}$. Consequently,

$$P_V \circ A(f) = P_V \circ A(f_1) + P_V \circ A(f_2) = A \circ P_V(f_1) + A \circ P_V(f_2) = A \circ P_V(f).$$

Lemma 2.5 Let U be an SI subspace of an SI space $V \subset L_2(\mathbb{R}^d)$. Then $len(U) \leq len(V)$.

Proof It is shown in [BDR2] that any SI subspace of $L_2(\mathbb{R}^d)$ is generated at most by a countable set. Thus, if $len(V) = \aleph_0$ we are done. Assume $V = S(\Phi)$ with $|\Phi| = len(V) = n$. Since U is SI, by Theorem 2.4 for any $\phi \in \Phi$ and $k \in \mathbb{Z}^d$, $P_U \circ S_k(\phi) = S_k \circ P_U(\phi)$. Since $U = P_U V = P_U S(\Phi) = S(P_U \Phi)$, U is an SI of length at most n.

Corollary 2.6 An SI subspace of a PSI space is PSI.

Corollary 2.7 An SI space $V \subset L_2(\mathbb{R}^d)$ is PSI if and only if every SI subspace of V is PSI.

Definition 2.8 The space $E_m(\mathbb{R}^d)$ is defined as the space of bounded measurable functions that decay faster than an inverse of a polynomial of a certain degree. To be exact,

$$E_m(\mathbb{R}^d) := \left\{ f \mid |f(x)| \leq C(1+|x|)^{-(m+d+\varepsilon)}, \text{ for some } \varepsilon > 0 \right\}.$$

Observe that $E_m(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$.

It is well known that Fourier techniques appear naturally in the analysis of SI spaces. Below is the definition of the Fourier transform we use in this work.

Definition 2.9 Let $f \in L_1(\mathbb{R}^d)$. Then the Fourier transform of f is defined by

$$\hat{f}(w) = \int_{\mathbb{R}^d} f(x) e^{-i\langle w, x \rangle} dx.$$

Properties of the Fourier transform we shall frequently use are presented below without proofs (see any introduction to Harmonic Analysis such as [K] for more details):

- $\hat{f} \in C(\mathbb{R}^d)$, $\|\hat{f}\|_{C(\mathbb{R}^d)} \le \|f\|_{L_1(\mathbb{R}^d)}$.
- Let $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ with $n_j \ge 0$, $j = 1, \ldots, d$.
  - If $f^{(k)} \in L_1(\mathbb{R}^d)$ for all $|k| \le |n|$, then $\widehat{f^{(n)}} = (i \cdot)^n \hat{f}$.
  - If $f \in E_m(\mathbb{R}^d)$ then $\hat{f}^{(n)} = (-i)^{|n|} \widehat{(\cdot)^n f}$, for all $|n| \le m$.
- The Fourier transform can be applied to functions in $L_2(\mathbb{R}^d)$. Furthermore, it is a one-one and onto mapping of $L_2(\mathbb{R}^d)$ into itself. For any $f, g \in L_2(\mathbb{R}^d)$ we have $\langle f, g \rangle = (2\pi)^{-d} \langle \hat{f}, \hat{g} \rangle$. In particular we have the **Plancherel-Parseval identity**

$$\|f\|^2_{L_2(\mathbb{R}^d)} = (2\pi)^{-d} \|\hat{f}\|^2_{L_2(\mathbb{R}^d)} . \tag{2.2}$$

The following is simple characterization of SI spaces in the Fourier domain.

Lemma 2.10 [BDR2] Let $S(\Phi)$ be an FSI space let $f \in L_2(\mathbb{R}^d)$. Then the following are equivalent:

1. $f \in S(\Phi)$.
2. There exist $\mathbb{T}^d$-periodic functions $\{\tau_\phi\}$ such that $\hat{f} = \sum_{\phi \in \Phi} \tau_\phi \hat{\phi}$.

We see that we can regard the generators of an FSI space as vectors spanning a finite dimensional vector space, with periodic functions playing the role of coefficients in the representations. Thus, we turn to Fourier based techniques. We begin by defining periodization tools that relate functions over $\mathbb{R}^d$ to periodic functions and facilitate analysis of the structure of shift invariant spaces.

Definition 2.11 The **bracket operator** $[\ ]: L_2(\mathbb{R}^d) \times L_2(\mathbb{R}^d) \to L_1(\mathbb{T}^d)$ is defined by

$$[f, g] = \sum_{k \in \mathbb{Z}^d} f(w + 2\pi k) \overline{g(w + 2\pi k)} .$$

For $f \in L_2(\mathbb{R}^d)$ the function $[f, f] \in L_1(\mathbb{T}^d)$ is called the **auto-correlation** of f.

Lemma 2.12 Let $f,g\in L_2(\mathbb{R}^d)$. Then for each $k\in\mathbb{Z}^d$, the k-th Fourier coefficient of $\left[\hat{f},\hat{g}\right]$ is

$$\left(\left[\hat{f},\hat{g}\right]\right)_k=\left\langle f,g(\cdot+k)\right\rangle,$$

from which we obtain the Fourier expansion

$$\left[\hat{f},\hat{g}\right](w)\sim\sum_{k\in\mathbb{Z}^d}\left\langle f,g(\cdot+k)\right\rangle e^{ikw}. \tag{2.3}$$

Proof For each $k\in\mathbb{Z}^d$ we compute the kth Fourier coefficient of $\left[\hat{f},\hat{g}\right]$

$$\begin{aligned}(2\pi)^{-d}\int_{\mathbb{T}^d}\left[\hat{f},\hat{g}\right]e^{-ikw}dw&=(2\pi)^{-d}\int_{\mathbb{T}^d}\left(\sum_{l\in\mathbb{Z}^d}\hat{f}(w+2\pi l)\overline{\hat{g}(w+2\pi l)}\right)e^{-ikw}dw\\&=(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{f}(w)\overline{\hat{g}(w)e^{ikw}}dw\\&=\left\langle f,g(\cdot+k)\right\rangle.\end{aligned}$$

◆

Observe that since for $f,g\in L_2(\mathbb{R}^d)$, $\left[\hat{f},\hat{g}\right]$ is in general only in $L_1(\mathbb{T}^d)$ and the Fourier series (2.3) may not converge (even in the L_1 norm [K]). To obtain pointwise convergence of (2.3) one must impose certain mild decay or smoothness conditions on the functions f,g and/or their Fourier transforms $\hat{f},\hat{g}$. We are mainly interested in cases where f,g are compactly supported for which there is an equality in (2.3) since $\left[\hat{f},\hat{g}\right]$ is a trigonometric polynomial.

Auto-correlations play a major role in our analysis. They are used in the definitions of stability constants, error kernels and "fine" error estimation constants. We require the following results on the convergence of auto-correlations. Let $m\geq 0$ and assume that $(\cdot)^m\rho_j\underset{L_2(\mathbb{R}^d)}{\to}(\cdot)^m\phi$.

Since $\hat{f}^{(m)}=(-i)^{|m|}\widehat{(\cdot)^m f}$, it follows from the continuity of the bracket operator that

$$\left[\hat{\rho}_j^{(m)},\hat{\rho}_j^{(m)}\right]\underset{L_1(\mathbb{T}^d)}{\to}\left[\hat{\phi}^{(m)},\hat{\phi}^{(m)}\right]. \tag{2.4}$$

For our analysis we require the stronger convergence

$$\left[\hat{\rho}_j^{(m)},\hat{\rho}_j^{(m)}\right]\underset{L_\infty(\mathbb{T}^d)}{\to}\left[\hat{\phi}^{(m)},\hat{\phi}^{(m)}\right], \tag{2.5}$$

which can be obtained by adding a condition on the functions' support.

Lemma 2.13 Assume that $\rho_j \underset{L_2(\mathbb{R}^d)}{\rightarrow} \phi$ so that $supp(\phi), supp(\rho_j) \subseteq \Omega$ where Ω is some bounded domain. Then for any $m \in \mathbb{Z}_+^d$ we have the convergence

$$\left[\hat{\rho}_j^{(m)}, \hat{\rho}_j^{(m)}\right] \underset{C(\mathbb{T}^d)}{\rightarrow} \left[\hat{\phi}^{(m)}, \hat{\phi}^{(m)}\right].$$

Proof It is easy to see that we also have $(\cdot)^m \rho_j \underset{L_2(\mathbb{R}^d)}{\rightarrow} (\cdot)^m \phi$ for any $m \in \mathbb{Z}_+^d$. By Lemma 2.12 we have that $\left[\hat{\phi}^{(m)}, \hat{\phi}^{(m)}\right]$, $\left[\hat{\rho}_j^{(m)}, \hat{\rho}_j^{(m)}\right]$ are trigonometric polynomials of uniformly bounded degree, say N. Therefore, the convergence of the Fourier coefficients

$$\left(\left[\hat{\rho}_j^{(m)}, \hat{\rho}_j^{(m)}\right]\right)_k = \left\langle (\cdot)^m \rho_j, (\cdot+k)^m \rho_j(\cdot+k)\right\rangle \underset{j\to\infty}{\rightarrow} \left\langle (\cdot)^m \phi, (\cdot+k)^m \phi(\cdot+k)\right\rangle = \left(\left[\hat{\phi}^{(m)}, \hat{\phi}^{(m)}\right]\right)_k,$$

for $|k| \le N$, implies the convergence (2.5).

♦

We now proceed to present "regularity" results for shift invariant spaces in $L_2(\mathbb{R}^d)$. The motivation to work with regular shift invariant spaces comes from applications where it is required to have a stable representation or approximation of signals. By stable we mean that small changes in an input function do not change much the representation and small changes in the representation change the reconstructed function only a little.

Definition 2.14 For each $f \in L_2(\mathbb{R}^d)$ we denote

$$\hat{f}_{\|w} := \left(\hat{f}(w + 2\pi k)\right)_{k\in\mathbb{Z}^d}, \qquad w \in \mathbb{T}^d.$$

Observe that

$$\left[\hat{f}, \hat{g}\right](w) = \left\langle \hat{f}_{\|w}, \hat{g}_{\|w}\right\rangle_{l_2(\mathbb{Z}^d)}, \quad w \in \mathbb{T}^d.$$

Let $S(\Phi)$ be an SI space. The **range** function associated with $S(\Phi)$ is

$$J_S(w) := span\left\{\hat{\phi}_{\|w} \mid \phi \in \Phi\right\} \tag{2.6}$$

The **spectrum** of $S(\Phi)$ is defined by

$$\sigma S(\Phi) := \left\{w \in \mathbb{T}^d \mid \dim J_S(w) > 0\right\}, \tag{2.7}$$

or equivalently

$$\sigma S(\Phi) := \left\{w \in \mathbb{T}^d \mid \left[\hat{\phi}, \hat{\phi}\right](w) \ne 0, \text{ for some } \phi \in \Phi\right\}.$$

Theorem 2.15 [BDR2] The range and spectrum of an SI space are invariants of the space. In particular they do not depend on the generating set.

Definition 2.16 Let S be an SI space. If $\dim J_S(w) \equiv const$, a.e., we say that S is **regular.**

Definition 2.17 Let $S(\Phi)$ be an FSI space. We say that Φ is a **basis** for S if for each $f \in S(\Phi)$ there are periodic functions τ_ϕ where $\hat{f} = \sum_{\phi\in\Phi} \tau_\phi \hat{\phi}$ and τ_ϕ are uniquely determined.

Observe that if $\mathbb{T}^d \setminus \sigma S(\Phi)$ is of positive measure then $S(\Phi)$ fails to be regular. In such a case $S(\Phi)$ does not have a basis. To see this we let Φ be any generating set. Choose any bounded periodic functions $\{\tau_\phi\}_{\phi\in\Phi}$, such that $\operatorname{supp}(\tau_\phi) \subseteq \mathbb{T}^d \setminus \sigma S(\Phi)$ and they are not all zero. Then,

$$\sum_{\phi\in\Phi} \tau_\phi \hat{\phi} = 0, \quad \text{a.e.}$$

Definition 2.18 Let $S(\Phi)$ be an SI space. The set Φ is called a **stable generating set** or a **stable basis** (for its span) if there exist constants $0 < A \le B < \infty$ such that for every $c = \{c_{\phi,k}\}_{\phi\in\Phi, k\in\mathbb{Z}^d} \in l_2(\Phi \times \mathbb{Z}^d)$

$$A\|c\|^2_{l_2(\Phi\times\mathbb{Z}^d)} \le \left\| \sum_{\phi\in\Phi, k\in\mathbb{Z}^d} c_{\phi,k}\phi(\cdot - k) \right\|^2_{L_2(\mathbb{R}^d)} \le B\|c\|^2_{l_2(\Phi\times\mathbb{Z}^d)}. \tag{2.8}$$

It can be shown ([BDR2]) that a stable basis is indeed a basis in the sense of Definition 2.17.

The next result is what motivates us to construct regular spaces.

Theorem 2.19 [BDR2] Let $S(\Phi)$ be an FSI space. Then $S(\Phi)$ is regular if and only if it contains a stable basis.

Observe that a (non trivial) PSI space $S(\phi)$ is regular if $\dim J_S(w) \equiv 1$ which is equivalent to $[\hat{\phi}, \hat{\phi}] > 0$ a.e. This ensures that any generator ϕ is a basis for $S(\phi)$. To see this, assume $\tau_\phi \hat{\phi} = 0$ for some periodic function τ_ϕ. Then, $|\tau_\phi|^2 [\hat{\phi}, \hat{\phi}] = 0$ which together with $[\hat{\phi}, \hat{\phi}] > 0$ implies $\tau_\phi = 0$. Nevertheless, the shifts of ϕ may fail to be a stable basis. For that we must have that the auto-correlation is bounded away from zero as the next well known result shows (see Theorem 2.3.6 in [RS1] for the general case of FSI spaces).

Theorem 2.20 [Me] Let $\phi \in L_2(\mathbb{R}^d)$ and $0 < A \le B < \infty$. Then the following are equivalent

1. $A\|c\|^2_{l_2(\mathbb{Z}^d)} \le \left\| \sum_{k\in\mathbb{Z}^d} c_k \phi(\cdot - k) \right\|^2_{L_2(\mathbb{R}^d)} \le B\|c\|^2_{l_2(\mathbb{Z}^d)}, \qquad \forall c \in l_2(\mathbb{Z}^d).$ (2.9)

2. $A \le \left[\hat{\phi},\hat{\phi}\right] \le B$, a.e. (2.10)

For example, if ϕ is compactly supported then its auto-correlation is a trigonometric polynomial and can only have a finite number of zeros in $\mathbb{T}^d$. In such a case $S(\phi)$ must be regular. If $\left[\hat{\phi},\hat{\phi}\right]$ has no zeros in $\mathbb{T}^d$, then by Theorem 2.20 ϕ is stable. Else we must "correct" ϕ, replacing it by a stable generator of $S(\phi)$, which must exist by Theorem 2.19. In the univariate case this "correction" can be obtained ([BDR2]) by selecting a unique (up to shifts and multiplications by constants) minimally supported generator of $S(\phi)$. However, in the multivariate case this "correction" produces, in general, a generator with infinite support. One can use the "orthogonalization trick" of [Da]: If $\left[\hat{\phi},\hat{\phi}\right] > 0$ a.e choose ψ by $\hat{\psi} = \hat{\phi}\Big/\left[\hat{\phi},\hat{\phi}\right]^{1/2}$. Then the shifts of ψ are an orthonormal basis for $S(\phi)$ and (2.8) holds with $A = B = 1$.

In refinable setting (see Definition 3.1) stability need only be ensured for one level, as this property is preserved by dilation. Let us see this for the PSI case. Assume $S(\phi) \subset S(\phi)^{1/2}$ and that ϕ is stable. Then for any dyadic dilation $S(\phi)^{2^{-j}} \subset S(\phi)^{2^{-(j+1)}}$ and the set $\left\{2^{jd/2}\phi(2^j \cdot - k)\right\}$ remains a stable basis for $S(\phi)^{2^{-j}}$ with the same stability constants of (2.8). Also, in such a case it is always possible ([BDR2]) to find an FSI space $S(\Psi)$ with a stable generating set $\Psi = \{\psi_l\}_{l=1}^{2^d-1}$ such that $S(\phi)^{2^{-j}} \oplus S(\Psi)^{2^{-j}} = S(\phi)^{2^{-(j+1)}}$. Thus, a stable decomposition exists at every scale. Also, under some conditions, the normalized set

$$\left\{2^{jd/2}\psi_l(2^j \cdot - k) \,\middle|\, 1 \le l \le 2^d - 1,\, j \in \mathbb{Z},\ k \in \mathbb{Z}^d\right\},$$

is a stable basis for $\overline{\bigcup_{j=-\infty}^{\infty} S(\phi)^{2^{-j}}}$ which is exactly $L_2(\mathbb{R}^d)$ if $S(\phi)$ provides any approximation order.

Whenever ϕ is stable but not refinable we still want to decompose the regular space $S(\phi)^{1/2}$ into a sum of two regular subspaces $S(\rho) \oplus S(\Psi) = S(\phi)^{1/2}$ where ρ is constructed to play the role of ϕ. First, we need tools to handle the following situation: Assume we have constructed a non regular FSI subspace $S(\Phi_m)$ of a regular FSI $S(\Phi_n)$ so that

$$len(S(\Phi_m)) = m < n = len(S(\Phi_n)).$$

We certainly can define $S(\Psi)$ as the orthogonal complement of $S(\Phi_m)$ in $S(\Phi_n)$ such that

$$S(\Phi_m)\oplus S(\Psi)=S(\Phi_n).$$

But the decomposition will have two undesirable features: Firstly, there is no choice of generators $\widetilde{\Phi_m},\widetilde{\Psi}$ such that $S(\widetilde{\Phi_m})=S(\Phi_m)$, $S(\widetilde{\Psi})=S(\Psi)$ and $\{\widetilde{\Phi_m},\widetilde{\Psi}\}$ is stable. Secondly, the decomposition may be somewhat redundant. Namely, $len(S(\Psi))>n-m$. We will show that this can be fixed by constructing $S(\Phi'_m)$ such that $S(\Phi_m)\subseteq S(\Phi'_m)\subset S(\Phi_n)$, $len(S(\Phi'_m))=m$ and $S(\Phi'_m)$ is regular. In doing so we ensure ([BDR2]) that the orthogonal complement is also regular and of length $n-m$. Hence, such a correction can produce a stable and efficient decomposition of $S(\Phi_n)$.

Lemma 2.21 Let $S(\phi)$ be a PSI space. Then there exists a regular PSI space $S(\phi')$ such that $S(\phi)\subseteq S(\phi')$.

Proof By definition $\sigma S(\phi)=supp(\tau)$ with $\tau:=\left[\hat{\phi},\hat{\phi}\right]$. Define $\psi\in L_2(\mathbb{R}^d)$ by $\hat{\psi}=\chi_{\mathbb{T}^d\setminus supp(\tau)}$. Next define $\phi'\in L_2(\mathbb{R}^d)$ as the inverse Fourier transform of $\hat{\phi}+\hat{\psi}$. Since $\chi_{\text{supp}(\tau)}\widehat{\phi'}=\hat{\phi}$ we have $S(\phi)\subseteq S(\phi')$. Next we show that $S(\phi')$ is regular. It is sufficient to show that $\mathbb{T}^d\setminus supp\left(\left[\widehat{\phi'},\widehat{\phi'}\right]\right)$ is of measure zero. Observe that our construction ensures that $\left[\hat{\phi},\hat{\psi}\right]=0$. Thus,

$$\begin{aligned}\left[\widehat{\phi'},\widehat{\phi'}\right]&=\sum_{k\in\mathbb{Z}^d}\left|\left(\hat{\phi}+\hat{\psi}\right)(\cdot+2\pi k)\right|^2\\&=\left[\hat{\phi},\hat{\phi}\right]+2Re\left[\hat{\phi},\hat{\psi}\right]+\left[\hat{\psi},\hat{\psi}\right]\\&=\left[\hat{\phi},\hat{\phi}\right]+\left[\hat{\psi},\hat{\psi}\right].\end{aligned}$$

We now see that $\left[\widehat{\phi'},\widehat{\phi'}\right]\neq 0$ a.e since

$$\left[\widehat{\phi'},\widehat{\phi'}\right](w)=\begin{cases}\tau(w) & w\in\text{supp}(\tau),\\ 1 & else.\end{cases}$$

Consequently, $S(\phi')$ is regular.

♦

Lemma 2.22 Let $S(\Phi)$ be a regular FSI space and let $\rho\in S(\Phi)$. Then there exists $\varphi\in S(\Phi)$, such that $S(\rho)\subseteq S(\varphi)$ and $S(\varphi)$ is a regular PSI subspace of $S(\Phi)$.

Proof If $S(\rho)$ is regular, we are done. Assume $S(\rho)$ fails to be regular. By Corollary 3.31 in [BDR2], we can assume the decomposition $S(\Phi)=\bigoplus_{i=1}^{n} S(\phi_i)$ such that each $S(\phi_i)$ is a (regular) PSI subspace and the shifts of ϕ_i are an orthonormal basis for $S(\phi_i)$. Therefore there exists a unique representation $\hat{\rho}=\sum_{i=1}^{n}\tau_i\hat{\phi}_i$ with τ_i periodic functions. Since $\left[\hat{\phi}_j,\hat{\phi}_k\right](w)=\delta_{j,k}$ for $1\le j,k\le n$ we have that $[\hat{\rho},\hat{\rho}]=\sum_{i=1}^{n}|\tau_i|^2$ and so $\sigma S(\rho)=\bigcup_{i=1}^{n}\operatorname{supp}(\tau_i)$. Define $\varphi\in S(\Phi)$ by

$$\hat{\varphi}=\tau_1'\hat{\phi}_1+\sum_{i=2}^{n}\tau_i\hat{\phi}_i, \qquad \tau_1'(w)=\begin{cases}1 & w\in\mathbb{T}^d\setminus\sigma S(\rho),\\ \tau_1(w) & else.\end{cases}$$

Then $[\hat{\varphi},\hat{\varphi}]=|\tau_1'|^2+\sum_{i=2}^{n}|\tau_i|^2$ and we can conclude the following

1. $S(\varphi)$ is regular since

$$\begin{aligned}\sigma S(\varphi)=\operatorname{supp}([\hat{\varphi},\hat{\varphi}])&=\operatorname{supp}(\tau_1')\cup\bigcup_{i=2}^{n}\operatorname{supp}(\tau_i)\\ &=\left(\mathbb{T}^d\setminus\sigma S(\rho)\right)\cup\operatorname{supp}(\tau_1)\cup\bigcup_{i=2}^{n}\operatorname{supp}(\tau_i)\\ &=\left(\mathbb{T}^d\setminus\sigma S(\rho)\right)\cup\sigma S(\rho)\\ &=\mathbb{T}^d.\end{aligned}$$

2. Evidently, $\hat{\rho}=\chi_{\sigma S(\rho)}\hat{\varphi}$ implies that $S(\rho)\subseteq S(\varphi)$.

♦

Theorem 2.23 Let U be a regular FSI. Then for any FSI subspace $S(\Phi_m)\subseteq U$ of length m there exists a regular subspace $S\left(\Phi_m'\right)$ of length m such that $S(\Phi_m)\subseteq S\left(\Phi_m'\right)\subseteq U$.

Proof The proof is essentially a Gram-Schmidt type construction, where we construct the "correction" $S\left(\Phi_m'\right)$ as an orthogonal sum of regular PSI spaces. We use induction on the length $|\Phi_m|=m$. The case $m=1$ follows by virtue of Lemma 2.22. Assume the claim is true for $m'<m$. Denote $\Phi_{m-1}=\{\phi_1,\ldots,\phi_{m-1}\}$ where $\Phi_m=\{\phi_1,\ldots,\phi_m\}$. Then by the induction hypothesis there exists a regular FSI subspace $S\left(\Phi_{m-1}'\right)$ such that

$$S(\Phi_{m-1})\subseteq S\left(\Phi_{m-1}'\right)\subset U,$$

and $len\left(S\left(\Phi_{m-1}{}'\right)\right)=\left|\Phi_{m-1}{}'\right|=m-1$. By [BDR2] the orthogonal complement in U of $S\left(\Phi_{m-1}{}'\right)$, denoted by W_{m-1}, is a regular FSI space. Let $S(\psi_m):=P_{W_{m-1}}S(\phi_m)$. Observe that $S(\psi_m)$ is not trivial since this would imply $S(\Phi_m)\subseteq S\left(\Phi_{m-1}{}'\right)$ which by Lemma 2.5 contradicts $len\left(S(\Phi_m)\right)=m$. Using again Lemma 2.22, we can find a regular PSI space $S\left(\phi_m{}'\right)$ such that

$$S(\psi_m)\subseteq S\left(\phi_m{}'\right)\subseteq W_{m-1}.$$

Since the orthogonal sum of two regular FSI spaces is regular, we have that $S\left(\Phi_m{}'\right)$, $\Phi_m{}'=\Phi_{m-1}{}'\cup\phi_m{}'$ is a regular FSI subspace of U. To conclude, observe that $S\left(\Phi_m{}'\right)$ also possesses the required properties of minimal length, $len\left(S\left(\Phi_m{}'\right)\right)=\left|\Phi_m{}'\right|=m$ and inclusion, $S(\Phi_m)\subseteq S\left(\Phi_m{}'\right)$.

♦

Next we discuss the special structure of the orthogonal projection into SI spaces.

Lemma 2.24 [BDR2] Let Φ be a basis for an FSI space $S(\Phi)$ and let $f\in L_2\left(\mathbb{R}^d\right)$. Then the orthogonal projection $P_{S(\Phi)}f$ is given by

$$\widehat{P_{S(\Phi)}f}=\sum_{\phi\in\Phi}\frac{\det G_{\hat{\phi}}\left(\hat{f}\right)}{\det G\left(\hat{\Phi}\right)}\hat{\phi}\quad, \tag{2.11}$$

where $G\left(\hat{\Phi}\right):=\left(\left[\hat{\phi},\hat{\psi}\right]\right)_{\phi,\psi\in\Phi}$ and $G_{\hat{\phi}}\left(\hat{f}\right)$ is obtained from $G\left(\hat{\Phi}\right)$ by replacing the ϕ-th row with $\left(\left[\hat{f},\hat{\psi}\right]\right)_{\psi\in\Phi}$.

In the PSI case the formula for the orthogonal projection (2.11) leads to the definition of the natural dual.

Definition 2.25 For any $\phi\in L_2\left(\mathbb{R}^d\right)$, the **(natural) dual** $\tilde{\phi}$ is defined by its Fourier transform

$$\hat{\tilde{\phi}}:=\frac{\hat{\phi}}{\left[\hat{\phi},\hat{\phi}\right]}, \tag{2.12}$$

where we interpret $0/0=0$.

From Lemma 2.12, it is easy to see that if the shifts of ϕ form an orthonormal basis for $S(\phi)$, then $\left[\hat{\phi},\hat{\phi}\right]=1$ a.e. and $\tilde{\phi}=\phi$. Equation (2.11) implies that in the PSI case $\widehat{P_{S(\phi)}f}=\left[\hat{f},\hat{\tilde{\phi}}\right]\hat{\phi}$. Transforming this back to the "time domain" we obtain the well known quasi-interpolation representation for the orthogonal projection

$$P_{S(\phi)}f=\sum_{k\in\mathbf{Z}^d}\left\langle f,\tilde{\phi}(\cdot-k)\right\rangle\phi(\cdot-k). \tag{2.13}$$

Frequently, in applications, one wishes to use compactly supported generators so that approximation algorithms are "local" and the complexity of the algorithms is minimal. This, for example, motivated the construction of the Daubechies' [Da] compactly supported orthonormal wavelets.

Definition 2.26 An FSI space V is called **local** if there exist a finite set of compactly supported functions, Φ, such that $V=S(\Phi)$.

Theorem 2.27 [BDR2] Any local FSI space is regular.

We require the following result on the special case of orthogonal projections of local SI spaces into local SI spaces.

Theorem 2.28 Let V,U be local FSI spaces. Then the orthogonal projection of V into U is a local FSI subspace. In particular it is a regular FSI space.

Proof Let $U=S(\Phi)$, $V=S(\Psi)$ be so that Φ,Ψ are a compactly supported generating sets for U,V respectively. Using Theorem 2.4, we have that $P_UV=P_US(\Psi)=S(P_U\Psi)$. Thus, it suffices to prove that for each $\psi\in\Psi$, there exists a compactly supported function $\psi'\in U$, such that $S(\psi')=S(P_U\psi)$. By virtue of (2.11) we have

$$\widehat{P_U\psi}=\sum_{\phi\in\Phi}\frac{\det G_{\hat{\phi}}(\hat{\psi})}{\det G\left(\hat{\Phi}\right)}\hat{\phi}. \tag{2.14}$$

By Lemma 2.12 the elements of the Gramian $G\left(\hat{\Phi}\right)$ are trigonometric polynomials. Thus, $\det G\left(\hat{\Phi}\right)$ is also a trigonometric polynomial so that $\det G\left(\hat{\Phi}\right)\neq 0$ a.e on $\mathbb{T}^d$. Let $\psi'\in S(P_U\psi)$ be defined by its Fourier transform $\widehat{\psi'}:=\det G\left(\hat{\Phi}\right)\widehat{P_U\psi}$. Then, the constructed generator ψ' has the required compact support property. Indeed, from (2.14) we have the representation $\widehat{\psi'}=\sum_{\phi\in\Phi}\det G_{\hat{\phi}}(\hat{\psi})\hat{\phi}$ where each $\det G_{\hat{\phi}}(\hat{\psi})$ is a trigonometric polynomial. This means that ψ'

can be expressed as a finite combination of compactly supported functions and therefore is compactly supported. To conclude, observe that since $\det G\left(\hat{\Phi}\right) \neq 0$ a.e, we have that $\widehat{P_U \psi} = \left(\det G(\Phi)\right)^{-1} \widehat{\psi'}$ and so $S(\psi') = S(P_U \psi)$.

♦

The following theorem is the main result of this chapter. It is a result on certain decompositions of FSI spaces with "good" approximation properties to an orthogonal sum of two FSI subspaces. Naturally, there are many ways to represent FSI spaces as a sum of two FSI subspaces, but our decomposition leads to "inheritance" of good approximation properties of the decomposed space by one of the subspaces. The underlying principal of the decomposition is "superfunction theory" ([BDR1]), which is explained in detail in Chapter 6.

Theorem 2.29 Let U_0 be a (local) regular FSI space of length $l_{U_0} \geq 2$. Let V be a (local) FSI space of length $1 \leq l_V < l_{U_0}$. Then U_0 can be decomposed $U_0 = U_1 \oplus W_1$ such that:

1. U_1 is a (local) regular FSI space of length $l_{U_1} = l_V$,
2. W_1 is a (local) regular FSI space of length $l_{W_1} = l_{U_0} - l_V$,
3. $W_1 \perp V$.

Proof

1. Let $\tilde{U}_1 := P_{U_0} V$. Observe that $\tilde{U}_1$ is an FSI subspace of U_0 with $len\left(\tilde{U}_1\right) \leq \min\left(l_{U_0}, l_V\right) = l_V$. Without loss of generality, $\tilde{U}_1$ is regular, else using Theorem 2.23, we can correct it to a regular subspace of U_0 of the same length. For the local case observe that by Theorem 2.28 we have that $\tilde{U}_1$ is local.
2. Since $\tilde{U}_1$ is (local) regular, by (Theorem 3.38) Theorem 3.13 in [BDR2] its orthogonal complement in U_0 denoted by $\tilde{W}_1$ is (local) regular and of length $l_{\tilde{W}_1} \geq l_{U_0} - l_V$. Let $\tilde{W}_1 = S\left(\psi_1, \ldots, \psi_{l_{\tilde{W}_1}}\right)$ such that $S(\psi_1, \ldots, \psi_i)$ is (local) regular for $1 \leq i \leq l_{\tilde{W}_1^\perp}$. We now define $W_1 := S\left(\psi_1, \ldots, \psi_{l_{W_1}}\right)$ where $l_{W_1} = l_{U_0} - l_V$. It is easy to see that $W_1 \perp V$.
3. We conclude the construction by setting U_1 as the orthogonal complement of W_1 in U_0. Using again (Theorem 3.38) Theorem 3.13 in [BDR2], U_1 is a (local) regular subspace of U_0 of length $l_{U_1} = l_{U_0} - l_{W_1} = l_V$.

♦

Example 2.30

1. Let ϕ, ψ be any known pair of univariate semi-orthogonal scaling function and wavelet, e.g., B-splines and B-wavelets ([Ch]). Define $U_0 = S(\phi)^{1/2}$ and $V = S(\phi)$. Then, the

refinability property, $S(\phi) \subset S(\phi)^{1/2}$, implies that the construction of Theorem 2.29 recovers the decomposition

$$S(\phi) \oplus S(\psi) = S(\phi)^{1/2}. \tag{2.15}$$

2. Let $S(\rho_0)$ be a univariate regular PSI space that is not refinable. Assume that ρ_0 provides L_2 approximation order m (see Definition 6.1). Select $U_0 = S(\rho_0)^{1/2}$, $V = S(\rho_0)$. Then the above construction finds a decomposition

$$S(\rho_1) \oplus S(\psi_1) = S(\rho_0)^{1/2}, \quad S(\psi_1) \perp S(\rho_0),$$

which in some sense mimics the refinable decomposition (2.15). Furthermore, we show in Section 6.4 that ρ_1 inherits the approximation order m from ρ_0 while the wavelet ψ_1 has m vanishing moments.

Finally, we wish to present another well known and useful periodization of functions over $\mathbb{R}^d$. In this case it is be more convenient to periodize functions to the unit cube. For any function $f \in L_1(\mathbb{R}^d)$ we define

$$\lambda_f := \sum_{k \in \mathbf{Z}^d} f(\cdot - k).$$

It is not too difficult to prove that $\lambda_f \in L_1([0,1]^d)$ and that $(\lambda_f)_k = \hat{f}(2\pi k)$ for each $k \in \mathbb{Z}^d$, where $(\lambda_f)_k$ is the k-th Fourier coefficient of λ_f. Thus, formally, **the Poisson Summation Formula** holds

$$\sum_{k \in \mathbf{Z}^d} f(x-k) = \sum_{k \in \mathbf{Z}^d} \hat{f}(2\pi k) e^{2\pi i k x}. \tag{2.16}$$

Assuming the Fourier series converges at the origin we obtain

$$\sum_{k \in \mathbf{Z}^d} f(k) = \sum_{k \in \mathbf{Z}^d} \hat{f}(2\pi k).$$

Again, since λ_f is in general only in $L_1([0,1]^d)$ the convergence may not take place even in the L_1 norm. Still, there are various decay and/or smoothness conditions one can impose on f such that there is pointwise convergence of (2.16). As we shall see in Section 6.2, the summation formula is a key ingredient in the theory of approximation from shift invariant spaces. In fact, in all previous work on this topic various different conditions were imposed on f such that the Fourier series of λ_f converges pointwise. For example, (2.16) holds pointwise if f is univariate, compactly supported, continuous and of bounded variation. We sometimes require pointwise convergence for a set of equations of type (2.16).

Definition 2.31 Let $f \in E_m(\mathbb{R}^d)$. We shall say that f satisfies the **summation condition of order** m if the Poisson Summation Formula (2.16) holds for all $(\cdot)^n f(x_0 - \cdot)$, $|n| < m$, $x_0 \in \mathbb{R}^d$.

First we observe the simple fact that if the Fourier series of λ_f converges pointwise, so will $\lambda_{f(x_0 -)}$ for any shift of f. Thus, the summation conditions are not an infinite set. Also, recall that for $f \in E_m(\mathbb{R}^d)$, the function $(\cdot)^n f$ is associated with a derivative of the Fourier transform by $\hat{f}^{(n)} = (-i)^{|n|}\widehat{(\cdot)^n f}$.

3 Multi-scale refinability

Definition 3.1 Let $S(\Phi)$ be an FSI space of $L_p(\mathbb{R}^d)$. The space S is said to be **refinable** if $S(\Phi) \subset S(\Phi)^{1/2}$. In such a case there exists a sequence of matrices $A_k \in M_{|\Phi|\times|\Phi|}(\mathbb{R})$ such that the following **two-scale relationship** holds

$$\Phi' = \sum_{k \in \mathbf{Z}^d} A_k \Phi(2\cdot -k)'. \tag{3.1}$$

Using the Fourier transform, an equivalent representation for (3.1) is

$$\hat{\Phi}' = P(2^{-1}\cdot)\hat{\Phi}(2^{-1}\cdot)', \qquad\qquad P(w) := 2^{-d}\sum_{k \in \mathbf{Z}^d} A_k e^{-ikw}. \tag{3.2}$$

If the products $\prod_{j=1}^{N} P(2^{-j}\cdot)$ converge, there exists distributional solution to the functional equation

$$\hat{\Phi} = \lim_{N\to\infty}\left(\prod_{j=1}^{N} P(2^{-j}\cdot)\right)\hat{\Phi}(2^{-N}\cdot)' = \left(\prod_{j=1}^{\infty} P(2^{-j}\cdot)\right)\hat{\Phi}(0)'. \tag{3.3}$$

For example, if $S(\phi)$ is PSI such that

$$\phi = \sum_{k \in \mathbf{Z}^d} p_k \phi(2\cdot -k),$$

then, assuming convergence, we have a representation

$$\hat{\phi} = \left(\prod_{j=1}^{\infty} P(2^{-j}\cdot)\right)\hat{\phi}(0), \qquad P(w) := 2^{-d}\sum_{k \in \mathbf{Z}^d} p_k e^{-ikw}.$$

We now present our generalization of two-scale refinability.

Definition 3.2 Let $S(\Phi)$ be an FSI space of $L_p(\mathbb{R}^d)$. We say that $S(\Phi)$ is **multi-scale (M – scale) refinable** if

$$S(\Phi) \subset S(\Phi)^{2^{-1}} + \cdots + S(\Phi)^{2^{-(M-1)}}, \quad \text{for some } M \in \mathbb{N},\ 2 \le M < \infty. \tag{3.4}$$

We see that for $M = 2$ we recover the well known two-scale refinability property. There is an important geometrical difference between the cases $M = 2$ and $M \geq 3$. If $S(\Phi_0)$ is multi-scale refinable with $M \geq 3$, then in general $S(\Phi_0) \not\subset S(\Phi_0)^{1/2}$. Thus, a (wavelet) decomposition of the type $S(\Phi_0)^{1/2} = S(\Phi_0) + S(\Psi)$ does not exist. As previously discussed, one of the objectives of this work is to overcome this obstacle by finding good decompositions of the type $S(\Phi_0)^{1/2} = S(\Phi_1) + S(\Psi)$ where $S(\Phi_1)$ plays the role of the low resolution of $S(\Phi_0)^{1/2}$. Nevertheless, we will shortly see another approach that constructs two-scale refinable spaces from multi-scale refinable spaces and thus allows the construction of classical stationary (multi) wavelets.

Let Φ be a generating set of a multi-scale refinable space. Then, there exist matrices $A_{m,k} \in \mathbb{M}_{|\Phi|\times|\Phi|}(\mathbb{R})$, $m = 1,\ldots,M-1$, $k \in \mathbb{Z}^d$ such that the following multi-scale relationship holds

$$\Phi' = \sum_{m=1}^{M-1} \sum_{k\in\mathbf{Z}^d} A_{m,k} \Phi\left(2^m \cdot - k\right)' . \tag{3.5}$$

In the PSI case the multi-scale relation takes the form

$$\phi = \sum_{m=1}^{M-1} \sum_{k\in\mathbf{Z}^d} p_{m,k} \phi\left(2^m \cdot - k\right), \tag{3.6}$$

where $P_m = \{p_{m,k}\}$, $m = 1,\ldots,M-1$ are given masks. Applying the Fourier transform to (3.5) we have

$$\hat{\Phi}' = \sum_{m=1}^{M-1} P_m\left(2^{-m} \cdot\right) \hat{\Phi}\left(2^{-m} \cdot\right)', \qquad P_m(w) := 2^{-md} \sum_{k\in\mathbf{Z}^d} A_{m,k} e^{-ikw} . \tag{3.7}$$

This general form yields a similar representation to (3.3). Assume that for $j \geq 0$ we have the representation

$$\hat{\Phi}(w) = \left(f^{j,M-1}(w),\ldots,f^{j+M-1,1}(w)\right)\left(\hat{\Phi}\left(2^{-j} w\right),\ldots,\hat{\Phi}\left(2^{-(j+M-1)} w\right)\right)',$$

where for $j = 0$ we have $\left(f^{0,M-1},\ldots f^{M-1,1}\right) = \left(\vec{1},\vec{0}\ldots,\vec{0}\right)$. Then by virtue of (3.7) we obtain a representation at the next scale $j+1$

$$\hat{\Phi}(w) = \left(f^{j+1,M-1}(w),\ldots,f^{j+M,1}(w)\right)\left(\hat{\Phi}\left(2^{-(j+1)} w\right),\ldots,\hat{\Phi}\left(2^{-(j+M)} w\right)\right)',$$

where

$$f^{j+m,M-m}(w) = \begin{cases} f^{j+m,M-m-1}(w) + P_m\left(2^{-(j+m)} w\right) f^{j,M-1}(\mathrm{w}) & 1 \leq m \leq M-2, \\ P_m\left(2^{-(j+m)} w\right) f^{j,M-1}(\mathrm{w}) & m = M-1. \end{cases}$$

We remark in passing that this is exactly the Fourier formulation of a multi-scale subdivision step that is detailed in the next chapter (see (4.2)). Therefore, in matrix notation with

$$P(w) := \begin{pmatrix} P_1(w) & P_2(2^{-1}w) & \cdots & \cdots & P_{M-1}\left(2^{-(M-1)}w\right) \\ I_{|\Phi|\times|\Phi|} & 0 & \cdots & 0 & 0 \\ 0 & I_{|\Phi|\times|\Phi|} & & \vdots & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & 0 & I_{|\Phi|\times|\Phi|} & 0 \end{pmatrix}, \tag{3.8}$$

we have formally

$$\begin{aligned} \hat{\Phi} &= \lim_{N\to\infty} \left(f^{N,M-1},\ldots,f^{N+M-1,1}\right)\left(\hat{\Phi}(2^{-N}\cdot),\ldots,\hat{\Phi}\left(2^{-(N+M-1)}\cdot\right)\right)' \\ &= \lim_{N\to\infty}\left(\left(\prod_{j=-N}^{-1} P(2^{j}\cdot)'\right)\left(\vec{1},\vec{0},\ldots,\vec{0}\right)'\right)'\left(\hat{\Phi}(2^{-N}\cdot),\ldots,\hat{\Phi}\left(2^{-(N+M-1)}\cdot\right)\right)' \\ &= \lim_{N\to\infty}\left(\vec{1},\vec{0},\ldots,\vec{0}\right)\left(\prod_{j=1}^{N} P(2^{-j}\cdot)\right)\left(\hat{\Phi}(2^{-N}\cdot),\ldots,\hat{\Phi}\left(2^{-(N+M-1)}\cdot\right)\right)' \\ &= \left(\vec{1},\vec{0},\ldots,\vec{0}\right)\left(\prod_{j=1}^{\infty} P(2^{-j}\cdot)\right)\left(\hat{\Phi}(0),\ldots,\hat{\Phi}(0)\right)'. \end{aligned} \tag{3.9}$$

Although our interest in the multi-scale functional equation (3.5) is primarily motivated by subdivision, the infinite product representation (3.9) can facilitate the analysis of existence of solutions to (3.5). It is interesting to note that the results of Heil and Colella in [HC], designed to handle two-scale matrix refinement equations are general enough to also deal with our multi-scale equations. The results from [HC] were later generalized in [CDP], using a different approach. Using the method of proof of [CDP], Theorem 3.2 one can show the following.

Theorem 3.3 Assume $a = (\vec{a},\ldots,\vec{a})$ is an eigenvector of $P(0)$ (defined by (3.8)) for the eigenvalue 1. Suppose that for some $\alpha > 0$, $P(w)$ satisfies

$$\|P(w) - P(0)\| \le C\|w\|^{\alpha},$$

and

$$\rho(P(0)) < 2^{\alpha},$$

where $\rho(A)$ denotes the spectral radius of A. Then

$$\hat{\Phi} = \left(\vec{1},\vec{0},\ldots,\vec{0}\right)\left(\prod_{j=1}^{\infty} P(2^{-j}\cdot)\right)(\vec{a},\ldots,\vec{a})',$$

is a solution to (3.5) in a distributional sense.

The following special case of the multi-scale functional equation arises naturally in multi-scale subdivision (see Theorem 4.4).

Example 3.4 Let us look at the three-scale functional equation

$$\phi=\sum_{k\in\mathbb{Z}^d} p_{1,k}\phi(2\cdot-k)+\sum_{k\in\mathbb{Z}^d} p_{2,k}\phi(4\cdot-k), \tag{3.10}$$

such that P_1, P_2 defined by (3.7) are trigonometric polynomials with the properties

$$\sum_{k\in\mathbb{Z}^d} p_{1,2k+\gamma}=C_1, \qquad \sum_{k\in\mathbb{Z}^d} p_{2,4k+\gamma}=C_2, \qquad C_1+C_2=1, \quad \gamma\in\mathbb{Z}^d.$$

It is easy to see that

$$P(0)=\begin{pmatrix} C_1 & C_2 \\ 1 & 0 \end{pmatrix},$$

and that $a=(1,1)$ is an eigenvector of the eigenvalue 1. Then, with the choice $\alpha=1$, Theorem 3.3 implies that if $-1<C_1<3$ there exists a compactly supported distributional solution to (3.10). ♦

Perhaps, the success of the approach of [HC], [CDP], when applied to the generalized multi-scale functional equation, is less surprising when one observes the following: If $S(\Phi)$ is M – scale refinable, in the sense of (3.4), we can merge the spaces $S(\Phi)^{2^{-m}}$, $m=0,\ldots,M-2$ and create a two-scale refinable FSI space $S(\Sigma)$ where

$$\Sigma=\bigcup_{m=0}^{M-2}\Phi_m, \quad \Phi_m:=\left\{\phi(2^m\cdot-k) \;\middle|\; \phi\in\Phi,\ k\in\{0,\ldots,2^m-1\}^d\right\}. \tag{3.11}$$

Indeed, $S(\Sigma)$ is two-refinable since

$$S(\Sigma)=S(\Phi)+\cdots+S(\Phi)^{2^{-(M-2)}}\subset S(\Phi)^{2^{-1}}+\cdots+S(\Phi)^{2^{-(M-1)}}=S(\Sigma)^{1/2}. \tag{3.12}$$

Although the above construction leads to classical two-scale refinability and solves geometrical difficulties, it has some undesirable features. We list below some of them:

1. There are (important) cases where Φ is a stable basis for $S(\Phi)$, but Σ fails to be a stable basis for $S(\Sigma)$ (see Theorem 4.12). Moreover, $S(\Sigma)$ may fail to be regular. Most of the present literature on analysis of refinable function vectors assumes stability of the generating

set (see [CDP] and the exact formula for the Sobolev smoothness index in [RS2], Theorem 3.14).
2. In Section 4.2 we show how (3.11) leads to the construction of two-scale matrix subdivision schemes from a multi-scale subdivision schemes. However, we shall see that the two-scale matrix representation is redundant. Furthermore, there are cases where multi-scale subdivision is meaningful and the implied two-scale matrix subdivision is not. This is the case with quasi convergent multi-scale schemes (see Definition 4.2).
3. Assume $S(\phi)$ is a three-scale refinable PSI space with some approximation order (see Section 6.2). Then, the constructed larger space $S(\Sigma)$, $\Sigma=\{\phi,\phi(2\cdot),\phi(2\cdot-1)\}$ is redundant from the viewpoint of "superfunction theory". That is, approximating functions using $S(\phi)^h$ is just as good (up to a constant) as approximating from the "larger" space $S(\Sigma)^h$ for any $h>0$.
4. Any wavelet basis $S(\Psi)$ that complements $S(\Sigma)$ such that $S(\Sigma)^{1/2}=S(\Sigma)+S(\Psi)$ is necessarily a multi-wavelet basis for the case $M>2$ (even in the univariate PSI case). This redundancy is more difficult to handle in applications such as signal compression.

Let $\phi\in L_1(\mathbb{R}^d)$. Using (3.2) it is easy to see that ϕ has a (finitely supported) two-scale relation if and only if $\hat{\phi}(2\cdot)/\hat{\phi}$ is a (trigonometric polynomial) 2π-periodic function. For example, let N_r be the univariate B-spline of order r. It is well known that $S(N_r)\subset S(N_r)^{1/2}$. Since the Fourier transform of N_r is

$$\hat{N}_r(w)=\left(\frac{1-e^{-iw}}{iw}\right)^r,$$

one can verify that $\hat{N}_r(2\cdot)/\hat{N}_r$ is indeed a trigonometric polynomial.

The following is a necessary condition for a function with a multi-scale relation to have a two-scale relation. While this condition is non-trivial, it uses only the given $M-1$ masks and does not require knowledge of the underlying function.

Lemma 3.5 Let $\phi\in L_1(\mathbb{R}^d)$ have an $M-$scale relation (3.6), where the masks are compactly supported. If ϕ has a two-scale relation, then there exists a trigonometric polynomial $\tau(w)$ which is the solution to the following equation

$$\prod_{m=1}^{M-1}\tau(2^{-m}w)=\sum_{m=1}^{M-1}\left(\prod_{j=m+1}^{M-1}\tau(2^{-j}w)\right)P_m(2^{-m}w). \tag{3.13}$$

Proof Assume ϕ has a two-scale relation with $\hat{\phi}(w)=\tau(2^{-1}w)\hat{\phi}(2^{-1}w)$. Then, we can expand the two-scale and $M-$ scale relations to obtain the following representations

$$\hat{\phi}(w)=\hat{\phi}\left(2^{-(M-1)}w\right)\prod_{m=1}^{M-1}\tau\left(2^{-m}w\right),$$

$$\hat{\phi}(w)=\sum_{m=1}^{M-1}P_m\left(2^{-m}w\right)\hat{\phi}\left(2^{-m}w\right)=\hat{\phi}\left(2^{-(M-1)}w\right)\sum_{m=1}^{M-1}\left(\prod_{j=m+1}^{M-1}\tau\left(2^{-j}w\right)\right)P_m\left(2^{-m}w\right).$$

Since $\phi\in L_1\left(\mathbb{R}^d\right)$, its Fourier transform is continuous and not identically zero. This implies that we can obtain (3.13) on some compact domain in $\mathbb{R}^d$. Since τ and $\{P_m\}$ are trigonometric polynomials, we can conclude that (3.13) holds for all $w\in\mathbb{R}^d$.

♦

Example 3.6

1. Let ϕ have a three-scale relation governed by P_1,P_2. By Lemma 3.5, if ϕ has a two-scale relation, there exists a trigonometric polynomial $\tau(w)$ such that

$$\tau(w)\tau(2w)=\tau(w)P_1(2w)+P_2(w).$$

2. Any two-scale refinable function satisfies infinitely many $M-$ scale functional equations. For example, assume $\phi\in L_2(\mathbb{R})$ with

$$\phi=\sum_{k=0}^{n}a_k\phi(2\cdot-k),\qquad \tau(w):=\frac{1}{2}\sum_{k=0}^{n}a_ke^{-ikw}.$$

We can construct from the above two-scale relation the following three-scale relation

$$\phi=\sum_{k=1}^{n}a_k\phi(2\cdot-k)+\sum_{k=0}^{n}a_0a_k\phi(4\cdot-k)=\sum_{k=1}^{n}p_{1,k}\phi(2\cdot-k)+\sum_{k=0}^{n}p_{2,k}\phi(4\cdot-k),$$

with $p_{1,k}=a_k$, $p_{2,k}=a_0a_k$. We see that τ solves (3.13) since

$$\left(\tau(2w)-P_1(2w)\right)\tau(w)=a_0\tau(w)=P_2(w).$$

♦

There are many examples for functions with good approximation properties which are multi-scale refinable but not two-scale refinable. An important family of such functions was constructed in [BTU]. Let $\phi=D(N_r)$ where D is any differential operator of degree $\le r-2$. It is easy to see that ϕ has a two-scale relation if and only if $D=I$. Thus, the function $OM_4=N_4+N_4''/42$ constructed in [BTU] is not two-scale refinable. Nevertheless, we now show that it has a three-scale relation of type (3.6).

Let ϕ be univariate, compactly supported and two-scale refinable, such that $\hat{\phi} = P(2^{-1})\hat{\phi}(2^{-1})$. Assume $\varphi = \sum_{n=1}^{M-1} \alpha_n \phi^{(d_n)}$ where $M \geq 2$, $\alpha_n \neq 0$, $n = 1,\ldots,M-1$ and $d_n \neq d_m$ for $n \neq m$ and ϕ sufficiently smooth. This implies that $\hat{\varphi}(w) = \hat{\phi}(w)\sum_{n=1}^{M-1} \beta_n w^{d_n}$ where $\beta_n = i^{d_n}\alpha_n$. For φ to be M – scale refinable, there should exist masks $P_m = \{p_{m,k}\}$, $m = 1,\ldots,M-1$, such that

$$\varphi = \sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}} p_{m,k}\varphi(2^m \cdot -k).$$

Assuming the existence of such masks gives

$$\hat{\phi}(w)\sum_{n=1}^{M-1}\beta_n w^{d_n} = \sum_{m=1}^{M-1}\sum_{n=1}^{M-1} 2^{-md_n}\beta_n w^{d_n} P_m(2^{-m}w)\hat{\phi}(2^{-m}w),$$

where $P_m(w) := 2^{-m}\sum_{k\in\mathbf{Z}} p_{m,k}e^{-ikw}$. Using the two-scale refinability of ϕ we obtain

$$\hat{\phi}(2^{-(M-1)}w)\prod_{j=1}^{M-1}P(2^{-j}w)\sum_{n=1}^{M-1}\beta_n w^{d_n} = \hat{\phi}(2^{-(M-1)}w)\sum_{n=1}^{M-1}\beta_n w^{d_n}\sum_{m=1}^{M-1}2^{-md_n}P_m(2^{-m}w)\prod_{j=1}^{M-m-1}P(2^{-m-j}w),$$

where products of type $\prod_{j=1}^{0}P(\cdot)$ are interpreted as 1. Since $\hat{\phi}$ is continuous and not identically zero, we obtain the following set of equations

$$\sum_{m=1}^{M-1}\left[2^{-md_n}\prod_{j=1}^{M-m-1}P(2^{-m-j}w)\right]P_m(2^{-m}w) = \prod_{j=1}^{M-1}P(2^{-j}w), \qquad n = 1,\ldots,M-1. \quad (3.14)$$

Example 3.7

1. The function $OM_4 = N_4 + N_4''/42$ constructed in [BTU] is three-scale refinable. To see this, let $P(w) = \frac{1}{16}(1+e^{-iw})^4$ be the two-scale relation of N_4. We seek masks P_1, P_2 that solve (3.14) with $M = 3$, $d_1 = 0$, $d_2 = 2$. Therefore the masks should satisfy

$$\begin{cases} P(w)P_1(2w) + P_2(w) = P(2w)P(w), \\ 4P(w)P_1(2w) + P_2(w) = 16P(2w)P(w). \end{cases}$$

One can easily verify the solution $P_1(w) = \frac{5}{16}(1+e^{-iw})^4$, $P_2(w) = -\frac{1}{64}(1+e^{-iw})^4(1+e^{-i2w})^4$.

2. It is easy to see that for each $n \in \mathbb{N}$, the function

$$\phi_n(x) = \begin{cases} 1 & x \in \left[-\frac{2n}{3}, \frac{2n}{3} \right], \\ 0 & else, \end{cases}$$

is three-scale refinable with

$$\phi_n(x) = \phi_n(2x) + \phi_n(4x + 2n) + \phi_n(4x - 2n).$$

♦

Next we show how from any multi-scale refinable function we can construct a multitude of multi-scale refinable functions using convolutions.

Theorem 3.8 Let $\phi \in L_1(\mathbb{R}^d)$ be M – scale refinable with masks $\{Q_m\}_{m=1}^{M-1}$ and $\rho \in L_1(\mathbb{R}^d)$ two-scale refinable with a corresponding mask P. Then $\phi * \rho$ is M – scale refinable with masks

$$P_m(w) = Q_m(w) \prod_{n=1}^{m} P(2^{m-n} w), \quad m = 1, \dots, M-1.$$

Proof The proof is a direct consequence of the property $\widehat{f * g} = \hat{f} \hat{g}$.

$$\begin{aligned} \widehat{\phi * \rho}(w) &= \sum_{m=1}^{M-1} Q_m(2^{-m} w) \hat{\phi}(2^{-m} w) \hat{\rho}(w) \\ &= \sum_{m=1}^{M-1} Q_m(2^{-m} w) \left(\prod_{n=1}^{m} P(2^{-n} w) \right) \hat{\phi}(2^{-m} w) \hat{\rho}(2^{-m} w) \\ &= \sum_{m=1}^{M-1} Q_m(2^{-m} w) \left(\prod_{n=1}^{m} P(2^{-n} w) \right) \widehat{\phi * \rho}(2^{-m} w). \end{aligned}$$

♦

Corollary 3.9 Let $\phi \in L_1(\mathbb{R})$ be M – scale refinable with masks $\{Q_m\}_{m=1}^{M-1}$. Then the function $\phi * N_r$ is M – scale refinable with masks

$$P_m(w) = Q_m(w) \prod_{n=1}^{m} P(2^{m-n} w), \quad P(w) = \left(\frac{1 + e^{-iw}}{2} \right)^r, \quad m = 1, \dots, M-1.$$

Furthermore, $\phi * N_r$ is in C^{r-2} and provides approximation order r (see Definition 6.1).

Thus, whenever we identify B-spline factors in the masks of a multi-scale refinable function we can immediately say something about its smoothness and approximation properties. This will become useful in Section 4.3 where we analyze multi-scale subdivision.

Finally we analyze the support size of multi-scale refinable functions. This is essential in the case where the functions do not have an analytic representation. Let ϕ be a univariate M – scale refinable function such that

$$\phi = \sum_{m=1}^{M-1} \sum_{k \in \mathbf{Z}} p_{m,k} \phi(2^m \cdot - k), \qquad P_m = \{p_{m,k}\},\ m = 1,\ldots,M-1. \tag{3.15}$$

Assume $\mathrm{supp}(\phi) = [a,b]$ and $\mathrm{supp}(P_m) \subseteq [\alpha_m, \beta_m]$ with $-\infty < \alpha_m < \beta_m < \infty$, $m = 1,\ldots,M-1$.
Using (3.15) it is clear that

$$a \geq \min_{1 \leq m \leq M-1} \left\{ \frac{a+\alpha_m}{2^m} \right\}, \quad b \leq \max_{1 \leq m \leq M-1} \left\{ \frac{b+\beta_m}{2^m} \right\}.$$

Therefore we obtain

$$a \geq \min_{1 \leq m \leq M-1} \left\{ \frac{\alpha_m}{2^m - 1} \right\}, \quad b \leq \max_{1 \leq m \leq M-1} \left\{ \frac{\beta_m}{2^m - 1} \right\},$$

which leads to

$$\mathrm{supp}(\phi) \subseteq \left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} \mathrm{supp}(P_m) \right\rangle, \tag{3.16}$$

where $\langle X \rangle$ denotes the convex hull of $X \subset \mathbb{R}$. In particular, ϕ has compact support whenever $\{P_m\}$ are finitely supported. In the next section we show an alternative approach that recovers (3.16) and also gives the same estimates for the support size in the multivariate case.

4 Multi-scale subdivision

The generalization of classical two-scale refinablity to multi-refinability naturally leads to a generalization of classical subdivision theory. We follow step by step the framework of two-scale subdivision ([Dy], [CDM]) and generalize the theory to multi-scale subdivision. As we shall see, the fact that multi-scale refinable spaces can be re-organized to a two-scale multiresolution (see the construction **(3.11)**) means that we can formulate multi-scale subdivision as special case of matrix subdivision. However, during this embedding we lose the structure presented below.

4.1 Multi-scale subdivision and the multi-scale refinable function

Definition 4.1 Let $m \ge 1$ and $d \ge 1$. We define $E_m^d := \{0,\dots,2^m-1\}^d$. For a vector $\gamma = (\gamma_1,\dots,\gamma_d) \in \mathbb{Z}^d$, we denote

$$\gamma\left(\operatorname{mod} E_m^d\right) := \left(\gamma_1\left(\operatorname{mod} 2^m\right),\dots,\gamma_d\left(\operatorname{mod} 2^m\right)\right).$$

The $M-$ scale subdivision algorithm uses $M-1$ subdivision masks $\{P_m\}_{m=1,\dots,M-1}$ with $P_m = \{p_{m,k}\}_{k\in\mathbb{Z}^d}$. They are used in the following convolutions

$$P_m * f := (P_m * f)_\alpha, \quad (P_m * f)_\alpha := \sum_{k\in\mathbb{Z}^d} p_{m,\alpha-2^m k} f_k, \quad f \in l_\infty(\mathbb{Z}^d). \tag{4.1}$$

Observe that the type of convolution used depends on the index of the mask $1 \le m \le M-1$.

Let $f^0 = \{f_k^0\}_{k\in Z^d}$ be initial given data. As usual we attach the value f_k^0 to the multi integer $k \in \mathbb{Z}^d$. Recall that a two-scale subdivision algorithm calculates the next level f^{j+1} from the current level f^j, using a single mask $P = \{p_k\}_{k\in\mathbb{Z}^d}$

$$f_\alpha^{j+1} = (P * f^j)_\alpha = \sum_{k\in\mathbb{Z}^d} p_{\alpha-2k} f_k^j,$$

thereby keeping one "active" level. The $M-$ scale algorithm uses $M-1$ "active" levels. We initialize the subdivision algorithm by setting the zero level $f^{0,M-1} = f^0$ and for $M \ge 3$ the next levels $f^{1,M-2} = \dots = f^{M-2,1} = \{0\}$. The first upper index corresponds to a refinement level, while the second to a state. In general the value $f_k^{j,m}$ corresponds to the level j at the state m,

$1 \le m \le M-1$ and is attached to the parameter $2^{-j}k$, $k \in \mathbb{Z}^d$. The iteration $j+1$ is defined using the convolutions (4.1)

$$f^{j+m,M-m} = \begin{cases} f^{j+m,M-m-1} + P_m * f^{j,M-1} & 1 \le m \le M-2, \\ P_m * f^{j,M-1} & m = M-1. \end{cases} \tag{4.2}$$

We see that an iteration calculates the final state of the level $j+1$, updates the temporary values at levels $j+2,\ldots,j+M-2$ (for $M>3$ only), adds the level $j+M-1$ and removes the level j from the "active" list. The algorithm can be terminated at any iteration j and its output is the sum of the $M-1$ "active" levels $j,\ldots j+M-2$. For large enough j, this sum approximates the limit of the scheme (see Definition 4.2)

$$F^j := \left(F^j\right)_k, \qquad \left(F^j\right)_k := \sum_{m=1}^{M-1} f_{2^{m-1}k}^{j+m-1,M-m}. \tag{4.3}$$

The approximation can be made continuous by any classical interpolation procedure. For example, let N_2 be the (linear) tensor product B-spline. Then at each level j we can construct the piecewise linear continuous sum

$$F^j(x) := \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} f_k^{j+m-1,M-m} N_2\left(2^{j+m-1}x - k\right). \tag{4.4}$$

Although we shall shortly present a possibly simpler description of multi-scale subdivision, we would like to draw a (natural) connection between the above representation and multi-scale refinability. Assume ϕ is $M-$scale refinable with

$$\phi = \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} p_{m,k}\phi\left(2^m \cdot - k\right). \tag{4.5}$$

Any $f \in S(\phi)$ (see Definition 2.2) has a representation (which need not be unique) using coefficients $f^{0,M-1} := \left\{f_k^{0,M-1}\right\}_{k \in \mathbb{Z}^d}$,

$$f = \sum_{k \in \mathbb{Z}^d} f_k^{0,M-1} \phi(\cdot - k). \tag{4.6}$$

Since also $f \in \sum_{m=1}^{M-1} S(\phi)^{2^{1-m}}$, for $M \ge 3$ the representation (4.6) can be expanded in a trivial way to

$$f = \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} f_k^{m-1,M-m} \phi\left(2^{m-1} \cdot - k\right),$$

with $f^{m-1,M-m} := \{f_k^{m-1,M-m}\} = 0$ for $2 \le m \le M-1$. Observe that this is equivalent to our initialization of the M – scale subdivision algorithm. By induction, assume that we have a representation of f in $\sum_{m=1}^{M-1} S(\phi)^{2^{-j-m+1}}$ for some scale $j \ge 0$, such that

$$f = \sum_{m=1}^{M-1} \sum_{k \in \mathbf{Z}} f_k^{j+m-1,M-m} \phi\left(2^{j+m-1} \cdot - k\right).$$

We now wish to increment this representation to the next finer dyadic scale and find a representation for f in $\sum_{m=1}^{M-1} S(\phi)^{2^{-j-m}}$. Using the multi-scale functional equation (4.5) we see that such a representation can be obtained exactly by the convolutions (4.1) and the (subdivision) iteration defined by (4.2)

$$f = \sum_{m=1}^{M-1} \sum_{k \in \mathbf{Z}^d} f_k^{j+m,M-m} \phi\left(2^{j+m} \cdot - k\right).$$

The relation between multi-scale subdivision and multi-scale refinable functions is established in Theorem 4.9.

Definition 4.2 An M – scale subdivision scheme $\mathcal{S}$ is said to be **convergent** if for any initial data of values $f^0 = \{f_k^0\}_{k \in Z^d}$ there exists a sum

$$f = \sum_{m=1}^{M-1} f_m, \qquad f_m \in C\left(\mathbb{R}^d\right), \tag{4.7}$$

such that for each dyadic point $2^{-J} k \in \mathbb{R}^d$

$$\lim_{J \le j \to \infty} \left| f_{2^{j-J}k}^{j,m} - f_m\left(2^{-J} k\right) \right| = 0, \quad 1 \le m \le M-1. \tag{4.8}$$

We also require that there exists some initial data f^0 for which the component f_{M-1} is not identically zero. The limit f is denoted by $\mathcal{S}^\infty f^0$. From the definition it is easy to see that if $\mathcal{S}$ is convergent, then the sum (4.3) also converges

$$\lim_{J \le j \to \infty} \left| F_{2^{j-J}k}^{j} - f\left(2^{-J} k\right) \right| = 0. \tag{4.9}$$

In cases where a decomposition of type (4.7) does not exist and component-wise convergence (4.8) fails to be true, but (4.9) holds for some $f \in C\left(\mathbb{R}^d\right)$, we say that $\mathcal{S}$ is **quasi convergent**.

The scheme $\mathcal{S}$ is **uniformly convergent (UC)** if for any $f^0 \in l_\infty(\mathbb{Z}^d)$ there exist continuous functions $f_1,\ldots,f_{M-1}$ such that

$$\lim_{j\to\infty}\sup_{k\in\mathbb{Z}^d}\left|f_k^{j,m} - f_m\left(2^{-j}k\right)\right| = 0\,, \qquad m = 1,\ldots,M-1\,. \tag{4.10}$$

Thus, with $f = \sum_{m=1}^{M-1} f_m$ we have

$$\lim_{j\to\infty}\sup_{k\in\mathbb{Z}^d}\left|F_k^j - f\left(2^{-j}k\right)\right| = 0\,. \tag{4.11}$$

In case (4.7) and (4.10) fail but (4.11) holds for some $f \in C\left(\mathbb{R}^d\right)$ we say that $\mathcal{S}$ is **uniformly quasi convergent (UQC)**.

Remarks

1. In this section convergence relates to the L_∞ norm. For weaker L_p – convergence of two-scale subdivision see for example [CDM] section 2.3.
2. We shall see that it is sufficient to impose conditions (4.8) and (4.10) only for $m = M-1$. The component-wise (uniform) convergence of the components $m = 1,\ldots,M-2$ is a consequence of the (uniform) convergence of the component $m = M-1$.
3. Observe that for the case of two-scale subdivision where $M = 2$ a scheme is UQC if and only if it is UC. Examples for UQC schemes that are not UC will be presented.

Next we describe an alternative implementation of the multi-scale subdivision algorithm that is useful. Assume $\mathcal{S}$ is an M – scale subdivision scheme and f^0 is initial data. This time we initialize the subdivision algorithm by $f^{-M+2,M-1} = \cdots = f^{-1,M-1} = 0$ for $M \geq 3$ and $f^{0,M-1} = f^0$. Then, each iteration $j \geq 1$ computes $f^{j,M-1}$ by

$$f_k^{j,M-1} = \sum_{m=1}^{M-1}\left(P_m * f^{j-m,M-1}\right)_k\,. \tag{4.12}$$

We see that in this implementation only the last state of each level is computed. Nevertheless, observe that at any level j we can obtain the values of the "full" scheme from the values of this partial scheme. By virtue of (4.2) we have

$$f_k^{j,m} = \sum_{r=1}^{m}\left(P_{M-r} * f^{j-(M-r),M-1}\right)_k\,, \qquad m = 1,\ldots,M-1\,. \tag{4.13}$$

Definition 4.3 Let $\mathcal{S}$ be an M – scale subdivision scheme. We denote by $\tilde{\mathcal{S}}$ the M – scale scheme that generates for each initial data f^0 only the levels $f^{j,M-1}$ using (4.12). We call $\tilde{\mathcal{S}}$ the

partial scheme of $\mathcal{S}$. We say that $\bar{\mathcal{S}}$ is **(uniformly) convergent** if for any initial data $f^0 := f^{0,M-1}$ the sequence $\left\{f^{j,M-1}\right\}_{j\geq 1}$ converges (uniformly).

Theorem 4.4 Let $\mathcal{S}$ be an M – scale subdivision scheme with masks $P_m = \left\{p_{m,k}\right\}$, $m = 1,\ldots,M-1$. Necessary conditions for $\mathcal{S}$ to converge uniformly are

$$1. \quad \sum_{k\in\mathbf{Z}^d} p_{m,2^m k+\gamma} = C_m, \qquad \text{where } m = 1,\ldots,M-1, \qquad \gamma \in E_m^d,$$

$$2. \quad \sum_{m=1}^{M-1} C_m = 1. \tag{4.14}$$

Proof By definition, there exists initial data f^0 such that $\mathcal{S}^\infty f^0 = \sum_{m=1}^{M-1} f_m$ with $f_m \in C\left(\mathbb{R}^d\right)$ and for some dyadic point $2^{-j_0}k_0$, $f_{M-1}\left(2^{-j_0}k_0\right) \neq 0$. We now use the partial scheme $\bar{\mathcal{S}}$ which produces only the sequences $f^{j,M-1}$. Using (4.12) we have for $j > j_0 + M$ and any $\gamma \in E_{M-1}^d$

$$f_{2^{j-j_0}k_0+\gamma}^{j,M-1} = \sum_{m=1}^{M-1}\left(P_m * f^{j-m,M-1}\right)_{2^{j-j_0}k_0+\gamma} = \sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}^d} p_{m,\gamma-2^m k} f_{k+2^{j-j_0-m}k_0}^{j-m,M-1}. \tag{4.15}$$

Since the masks $\left\{P_m\right\}$ have finite support and $f^{j,M-1}$ converges uniformly to $f_{M-1} \in C\left(\mathbb{R}^d\right)$, there exists $J(\Omega,\varepsilon) \geq j_0$ such that for $j > J + M$ we have locally $f_\alpha^{j-m,M-1} \approx f_{M-1}\left(2^{-j_0}k_0\right) \neq 0$, $m = 0,\ldots,M-1$. Thus, we can derive from (4.15)

$$\left|1 - \sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}^d} p_{m,\gamma-2^m k}\right| \leq A\varepsilon, \qquad \gamma \in E_{M-1}^d,$$

where A is a constant which depends on the sum of the support sizes of $\left\{P_m\right\}$ and $\varepsilon > 0$ is arbitrarily small. We conclude that

$$\sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}^d} p_{m,2^m k+\gamma} = 1, \quad \gamma \in E_{M-1}^d.$$

It remains to show that

$$\sum_{k\in\mathbf{Z}^d} p_{m,2^m k+\gamma_m} = C_m, \qquad m = 1,\ldots,M-1, \qquad \gamma_m \in E_m^d.$$

We now use the scheme $\mathcal{S}$. Observe that for $M=2$ we are done and so we can assume that $M \geq 3$. For any $1 \leq m \leq M-2$ and $j \geq j_0 + M$ we have

$$f_\alpha^{j+m,M-m} = f_\alpha^{j+m,M-m-1} + \sum_{k \in \mathbf{Z}^d} p_{m,\alpha - 2^m k} f_k^{j,M-1}.$$

Let $\alpha = 2^{j-j_0} k_0 + \gamma_m$, with $\gamma_m \in E_m^d$. Observe that $\alpha \equiv \gamma_m \left(\operatorname{mod} E_m^d\right)$. Since the scheme is UC, this implies that for big enough $j > j_0$ we have locally about the point $2^{-j_0} k_0$

$$f_{M-m}\left(2^{-j_0} k_0\right) \approx f_{M-m-1}\left(2^{-j_0} k_0\right) + f_{M-1}\left(2^{-j_0} k_0\right) \sum_{k \in \mathbf{Z}^d} p_{m,2^m k + \gamma_m}.$$

Thus, since $f_{M-1}\left(2^{-j_0} k_0\right) \neq 0$ we obtain for $1 \leq m \leq M-2$

$$\sum_{k \in \mathbf{Z}^d} p_{m,2^m k + \gamma_m} = \frac{f_{M-m} - f_{M-m-1}}{f_{M-1}}\left(2^{-j_0} k_0\right), \qquad \gamma_m \in E_m^d.$$

Together with the first part of the proof, this also implies that $\sum_{k \in \mathbf{Z}^d} p_{M-1,2^{M-1}k+\gamma} = C_{M-1}$, $\gamma \in E_{M-1}^d$.

♦

Corollary 4.5 Assume the masks $\{P_m\}$ of a UQC $M-$scale scheme meet conditions (4.14). If the masks also meet the following additional conditions (up to a shift):

$$p_{m,2^m k} = \begin{cases} C_m & k = 0, \\ 0 & else, \end{cases} \qquad m = 1,\ldots,M-1,$$

then the scheme is interpolatory.

Proof We shall see that the above conditions ensure that once "sums" are computed at some refinement level, their value is retained at higher levels. Namely, we prove that for any $j \geq 0$ and $\alpha \in \mathbb{Z}^d$

$$F_{2\alpha}^{j+1} = F_\alpha^j.$$

The proof is by direct computation

$$\begin{aligned} F_{2\alpha}^{j+1} = \sum_{m=1}^{M-1} f_{2^{m-1}2\alpha}^{j+m,M-m} &= \sum_{m=1}^{M-2}\left(f_{2^{m-1}2\alpha}^{j+m,M-m-1} + \left(P_m * f^{j,M-1}\right)_{2^{m-1}2\alpha}\right) + \left(P_{M-1} * f^{j,M-1}\right)_{2^{m-1}2\alpha} \\ &= \sum_{m=1}^{M-2} f_{2^{m-1}2\alpha}^{j+m,M-m-1} + \sum_{m=1}^{M-1}\sum_{k \in \mathbf{Z}^d} p_{m,2^{m-1}2\alpha - 2^m k} f_k^{j,M-1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{M-2} f_{2^m \alpha}^{j+m,M-m-1} + \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} p_{m,2^m(\alpha-k)} f_k^{j,M-1} \\
&= \sum_{m=1}^{M-2} f_{2^m \alpha}^{j+m,M-m-1} + \sum_{m=1}^{M-1} C_m f_\alpha^{j,M-1} \\
&= \sum_{m=0}^{M-2} f_{2^m \alpha}^{j+m,M-m-1} \\
&= F_\alpha^j .
\end{aligned}
$$

Consequently, if the scheme is UQC we obtain that for the limit function $f = \mathcal{S}^\infty \{f_k^0\}$ of any initial data $\{f_k^0\}$ we have that $f(k) = f_k^0$, $k \in \mathbb{Z}^d$.

◆

Example 4.6

1. It is easy to see that for $M = 2$, the necessary conditions (4.14) recover the classical two-scale subdivision necessary conditions

$$
\sum_{k \in \mathbb{Z}^d} p_{1,2k+\gamma} = 1, \qquad \gamma \in E_1^d .
$$

2. Let us define the following family of univariate three-scale subdivision schemes

$$
\left\{ \mathcal{S}_{3,\beta} \middle| \, \beta \in \mathbb{R} \right\}.
$$

Each member of $\{\mathcal{S}_{3,\beta}\}$ is defined by its masks

$$
P_{\beta,1}(z) = \frac{\beta}{2} z^{-1} (1+z)^2, \qquad P_{\beta,2}(z) = \frac{1-\beta}{16} z^{-4} (1+z)^4 \left(1+z^2\right)^2 .
$$

Observe that for the choice $\beta = 1$, we obtain the two-scale scheme corresponding to the linear B-spline. The family $\{\mathcal{S}_{3,\beta}\}$ has the following properties:

a. The schemes $\{\mathcal{S}_{3,\beta}\}$ satisfy the necessary conditions (4.14) with $C_1 = \beta$, $C_2 = 1 - \beta$.
b. We will prove (Example 4.21) that for a certain range of β, the scheme is UQC and C^1. For example the scheme is smooth for the choice $\beta = 1/2$. In such a case the centered masks are:

$$
P_{\frac{1}{2},1} = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\}, \qquad P_{\frac{1}{2},2} = \left\{ \frac{1}{32}, \frac{4}{32}, \frac{8}{32}, \frac{12}{32}, \frac{14}{32}, \frac{12}{32}, \frac{8}{32}, \frac{4}{32}, \frac{1}{32} \right\}.
$$

The above scheme is also "almost interpolating", shape-preserving and reproduces polynomials of degree one. Furthermore, numeric simulation shows that for the choice $\beta = 1/2$ the scheme is UC.

c. For a UQC scheme $\mathcal{S} \in \{\mathcal{S}_{3,\beta}\}$ the corresponding $\mathcal{S}$ – refinable function (see Theorem 4.9) has support in $[-4/3, 4/3]$. Recall that for the special case of two-scale subdivision, there are known sharp bounds on the smoothness of the scheme using the support size of the mask (see [CDM] Corollary 2.1 and [DaL] Theorem 5.1). In this sense the above scheme is optimal.

♦

Next we show that in the case of uniform convergence the requirement for component-wise uniform convergence tightly couples the continuous components $\{f_m\}_{m=1,\dots,M-1}$ of the limit $f = \mathcal{S}^\infty f^0$. In fact, they are identical up to a multiplicative constant.

Lemma 4.7 Let $M \ge 2$ and assume $\{P_m\}_{m=1,\dots,M-1}$ are masks for which conditions (4.14) hold. Then the corresponding M – scale scheme $\mathcal{S}$ is UC if and only if the corresponding partial scheme $\tilde{\mathcal{S}}$ is UC.

Proof If $\mathcal{S}$ is UC then because $\tilde{\mathcal{S}}$ generates the levels $f^{j,M-1}$, it is by definition UC. We now assume that $\tilde{\mathcal{S}}$ is UC. Since the necessary conditions (4.14) hold we have for any $1 \le m \le M-1$ and $\gamma \in \mathbb{Z}^d$

$$\sum_{k \in \mathbb{Z}^d} p_{m,2^m k+\gamma} = C_m .$$

In the limit we obtain from (4.13) for each dyadic point $2^{-j_0} k$

$$\begin{aligned}\lim_{j_0 < j \to \infty} f^{j,m}_{2^{j-j_0}k} &= \lim_{j_0 < j \to \infty} \sum_{r=1}^{m} \left(P_{M-r} * f^{j-(M-r),M-1} \right)_{2^{j-j_0}k} \\ &= f_{M-1}\left(2^{-j_0} k\right) \sum_{r=1}^{m} C_{M-r},\end{aligned}$$

where we have used sufficiently small neighborhoods of $2^{-j_0} k$ for each $j \to \infty$.

♦

Corollary 4.8 Let $\mathcal{S}$ be a UC M – scale subdivision scheme. Then the masks of $\mathcal{S}$ satisfy (4.14) and for any initial data f^0 such that $\mathcal{S}^\infty f^0 = f = \sum_{m=1}^{M-1} f_m$, $f_m \in C\left(\mathbb{R}^d\right)$ we have

$$f_m(x) = \left(\sum_{r=1}^{m} C_{M-r} \right) f_{M-1}(x), \qquad m = 1, \dots, M-1,$$

and

$$f(x) = \left(\sum_{m=1}^{M-1} m C_m \right) f_{M-1}(x). \tag{4.16}$$

Remark In view of Corollary 4.8 we can replace in applications the full scheme by the partial scheme. If we terminate the partial algorithm at the level j, we can estimate the limit of the full scheme $f = \mathcal{S}^\infty f^0$ by

$$F_k^j \approx \left(\sum_{m=1}^{M-1} m C_m \right) f_k^{j,M-1}.$$

We now connect our two generalizations, multi-scale refinability (see Definition 3.2) and multi-scale subdivision.

Theorem 4.9 Let $\mathcal{S}$ be a UQC M – scale subdivision scheme for which the necessary conditions (4.14) hold. Then its finitely supported masks $\{P_m\}$ determine a unique compactly supported function $\phi \in C(\mathbb{R}^d)$ with the following properties

1. (M – scale relation) $$\phi(x) = \sum_{m=1}^{M-1} \sum_{k \in \mathbf{Z}^d} p_{m,k} \phi(2^m x - k), \tag{4.17}$$

2. (Partition of unity) $$\sum_{k \in \mathbf{Z}^d} \phi(\cdot - k) = 1. \tag{4.18}$$

Furthermore, ϕ has the following properties

3. (Compact support) $$\operatorname{supp}(\phi) \subseteq \left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} \operatorname{supp}(P_m) \right\rangle, \tag{4.19}$$

where for any set $X \subset \mathbb{R}^d$, $\langle X \rangle$ is the convex hull of X.

4. ($\mathcal{S}$ – refinable function) For any initial data f^0 we have that

$$\mathcal{S}^\infty f^0 = \sum_{k \in \mathbf{Z}^d} f_k^0 \phi(\cdot - k). \tag{4.20}$$

Proof We select the initial data $f^0 := \{\delta_{0,k}\}$ and denote $\mathcal{S}^\infty f^0 = \phi \in C(\mathbb{R})$. First we establish the compact support property (4.19). Since by (4.13) we have that $\operatorname{supp}(\phi) \subseteq \lim_{j \to \infty} \operatorname{supp}(f^{j,M-1})$ it is sufficient to consider the partial scheme.

Remark Observe that since we only assumed that the scheme is UQC, it is possible that the sequence $\{f^{j,M-1}\}$ diverges. Nevertheless, any bound we obtain on the support of the partial scheme can serve as a bound for the support of ϕ.

We initialize the partial scheme with $f^{2-M,M-1} = \cdots = f^{-1,M-1} = 0$ for $M \geq 3$ and $f^{0,M-1} := \{\delta_{0,k}\}$. From (4.13) we have that

$$f_a^{j,M-1} = \sum_{m=1}^{M-1} \left(P_m * f^{j-m,M-1}\right)_a = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,a-2^m k} f_k^{j-m,M-1} . \tag{4.21}$$

We claim that for $j \ge 2-M$

$$supp\left(f^{j,M-1}\right) \subseteq \left(2^j-1\right)\left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m-1} supp\left(P_m\right)\right\rangle . \tag{4.22}$$

We prove (4.22) by induction on the refinement levels. For $j = 2-M,\ldots,-1$, we have that $supp\left(f^{j,M-1}\right) = \varnothing$. For $j=0$, we have that $supp\left(f^0\right) = supp\left(\delta_{k,0}\right) = \{0\}$. Also observe that $f_k^{1,M-1} = p_{1,k}$ and so (4.22) holds for all $j = 2-M,\ldots,1$. Assume by induction that (4.22) holds for all $j' < j$ with $j > 1$. From (4.21) we can see that for each $m = 1,\ldots,M-1$, the contribution to the support of $f^{j,M-1}$ of the convolution $P_m * f^{j-m,M-1}$ is contained in

$$2^m supp\left(f^{j-m,M-1}\right) + supp\left(P_m\right) \subseteq \left(2^j - 2^m\right)\left\langle \bigcup_{r=1}^{M-1} \frac{1}{2^r-1} supp\left(P_r\right)\right\rangle + supp\left(P_m\right),$$

where the sums are Minkowski sums of sets in $\mathbb{R}^d$. Let us define the sets $X_m := \frac{1}{2^m-1} supp\left(P_m\right)$, $m = 1,\ldots,M-1$. Then

$$\left(2^j-2^m\right)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle + \left(2^m-1\right)X_m \subseteq \left(2^j-2^m\right)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle + \left(2^m-1\right)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle = \left(2^j-1\right)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle .$$

We can now derive (4.22) since

$$supp\left(f^{j,M-1}\right) \subseteq \left(2^j-1\right)\left\langle \bigcup_{m=1}^{M-1} X_m\right\rangle = \left(2^j-1\right)\left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m-1} supp\left(P_m\right)\right\rangle .$$

Therefore, since the value of $f_k^{j,M-1}$ is attached to the parameter $2^{-j}k$, we obtain

$$supp(\phi) \subseteq \lim_{j\to\infty} 2^{-j}\left\langle supp\left(f^{j,M-1}\right)\right\rangle \subseteq \left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m-1} supp\left(P_m\right)\right\rangle ,$$

and so (4.19) holds. Since ϕ has compact support we can use the linearity of the subdivision operator $\mathcal{S}$ to obtain for any initial data the representation (4.20).

Next we verify (4.17). It is easy to see that after the first iteration of the subdivision process the "active" levels are $1,\ldots,M-1$ with

$$\left(f^{m,M-m}\right)_k = p_{m,k}, \qquad 1 \le m \le M-1. \tag{4.23}$$

We now separate the levels of (4.23) and define for each m_0, $1 \le m_0 \le M-1$, initial control points $g^{m,M-m}$, $1 \le m \le M-1$

$$\left(g^{m,M-m}\right)_k := \begin{cases} p_{m_0,k} & m = m_0, \\ 0 & else. \end{cases}$$

After $m_0 - 1$ iterations of the M – scale subdivision algorithm on this initial data, the "active" levels are $m_0, \ldots, m_0 + M - 1$. It is easy to see that for $M \ge 3$, the active levels $m_0 + 1, \ldots, m_0 + M - 2$ are zero while for the first "active" level we have

$$\left(g^{m_0,M-1}\right)_k = p_{m_0,k}.$$

By dilating (4.20) we obtain that the limit for the initial data $g^{m,M-m}$, $1 \le m \le M-1$ is

$$\sum_{k \in \mathbf{Z}^d} p_{m_0,k} \phi\left(2^{m_0} \cdot - k\right). \tag{4.24}$$

Using again the compact support of ϕ and the linearity of the scheme, we can sum up the limits (4.24) to obtain (4.17).

To prove that ϕ has the partition of unity property (4.18), we choose $f^0 \equiv 1$. Since the subdivision algorithm is initialized by $f^{1,M-2} = \cdots = f^{M-2,1} = 0$, the initial sum of levels is $F^0 \equiv 1$. Assume by induction that after iteration j each of the "active" levels $j, \ldots, j+M-2$ is constant, $f^{j+m,M-m-1} \equiv \alpha_{j+m}$ such that $\sum_{m=0}^{M-2} \alpha_{j+m} = 1$ implying that $F^j \equiv 1$. Since we assumed that conditions (4.14) hold, it is easy to see that after iteration $j+1$, the "active" levels are $j+1, \ldots, j+M-1$ with

$$f^{j+m,M-m} \equiv \begin{cases} \alpha_{j+m} + C_m \alpha_j & 1 \le m \le M-2, \\ C_{M-1} \alpha_j & m = M-1. \end{cases}$$

Summing up the "active levels" we see that

$$\begin{aligned} F^{j+1} &\equiv C_{M-1}\alpha_j + \sum_{m=1}^{M-2} \left(\alpha_{j+m} + C_m \alpha_j\right) \\ &= \alpha_j \sum_{m=1}^{M-1} C_m + \sum_{m=1}^{M-2} \alpha_{j+m} \\ &= \sum_{m=0}^{M-2} \alpha_{j+m} = 1. \end{aligned}$$

Thus,

$$\sum_{k\in\mathbb{Z}^d}\phi(\cdot-k)=\sum_{k\in\mathbb{Z}^d}f_k^0\phi(\cdot-k)=S^\infty f^0=\lim_{j\to\infty}F^j\equiv 1.$$

Finally, the uniqueness of a generating function satisfying both (4.17) and (4.18) can be proved using the same arguments used in [Dy] for the two-scale case.

♦

Remark In the proof of the last theorem, we used a rather complex argument to bound the support size of the $\mathcal{S}$ – refinable function $\phi=\mathcal{S}^\infty\delta$. Recall that we have already presented the bound (4.19) using a simpler approach in Chapter 3. However, here the bound is used to derive the multi-scale relation.

The above result shows the connection between multi-scale subdivision schemes and multi-scale refinability: for each UQC multi-scale scheme there exists a corresponding continuous $\mathcal{S}$ – **refinable function** for which a multi-scale relation holds using the scheme's masks. As explained in the next section this also leads to a relation between multi-scale subdivision and matrix subdivision.

We conclude this section by presenting methods to compute the $\mathcal{S}$ – refinable function $\phi=\mathcal{S}^\infty\delta$. Obviously, we can use the method of proof of Theorem 4.9, initialize the subdivision algorithm with $\{\delta_{k,0}\}$ and converge to ϕ. Observe that for schemes that are only UQC this approach can be unstable. Namely, while the sum of the refinement levels converges to ϕ, the components themselves can "blow-up". An alternative approach, which is known to work well in two-scale subdivision, is to first compute the values $\phi(k)$, $k\in\mathbb{Z}^d$ and then use the multi-scale relation (4.17) to recursively compute values at finer dyadic points. For example, assuming the values $\phi(k)$, $k\in\mathbb{Z}^d$ are known, the first iteration that produces the values at the half-integers is

$$\phi\left(2^{-1}n\right)=\sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}^d}p_{m,k}\phi\left(2^m2^{-1}n-k\right)=\sum_{m=0}^{M-2}\sum_{k\in\mathbb{Z}^d}p_{m+1,k}\phi\left(2^m n-k\right),\qquad n\in\mathbb{Z}^d. \tag{4.25}$$

Since ϕ has compact support, only a finite number of values $\left\{\phi\left(2^m n-k\right)\right\}$ in (4.25) is non-zero. Thus, we only need to describe how the values at the points $k\in\mathbb{Z}^d$ can be computed directly from the masks $\{P_m\}$. We first observe that the case of $\phi(k)=0$ for all $k\in\mathbb{Z}^d$ is not possible. This implies by the above recursive method that $\phi\left(2^{-j}k\right)=0$ for any dyadic point $2^{-j}k$ with $j\ge 0$, $k\in\mathbb{Z}^d$ which by continuity leads to $\phi(x)=0$ for all $x\in\mathbb{R}^d$. Thus, for the finite set $\Lambda:=\left\{\alpha\middle|\ \alpha\in supp(\phi)\cap\mathbb{Z}^d\right\}$, the value set $\left\{\phi(\alpha)\middle|\ \alpha\in\Lambda\right\}$ is not trivial. By assuming some order on Λ, we can define the vector $V_\phi\in\mathbb{R}^{|\Lambda|}$, $V_\phi:=\left(\phi(\alpha)\right)_{\alpha\in\Lambda}$. Also, for each $\alpha\in\Lambda$ we have that,

$$\phi(\alpha)=\sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}^d}p_{m,k}\phi\left(2^m\alpha-k\right)=\sum_{m=1}^{M-1}\sum_{\beta\in\Lambda}p_{m,2^m\alpha-\beta}\phi(\beta)=\sum_{\beta\in\Lambda}\phi(\beta)\sum_{m=1}^{M-1}p_{m,2^m\alpha-\beta},$$

which naturally leads to the construction of the matrix $A \in \mathbb{M}_{|\Lambda|\times|\Lambda|}(\mathbb{R})$

$$A := \left(\sum_{m=1}^{M-1} p_{m,2^m \alpha - \beta} \right)_{\alpha,\beta\in\Lambda}.$$

It is easy to see that $V_\phi A = V_\phi$. Since V_ϕ is not trivial, we can conclude that the values of the $\mathcal{S}$ – refinable function at the integers correspond to a left eigenvector of the matrix A with eigenvalue 1. By the discussion above there is a one-to-one correspondence between vectors V_ϕ and $\mathcal{S}$ – refinable functions ϕ. Therefore the uniqueness of the generating function ϕ up to a multiplicative factor implies that the eigenvector subspace corresponding to the eigenvalue 1 is of dimension 1.

Following [CDM], this approach can be generalized in the following way. If ϕ is known to be in $C^m(\mathbb{R}^d)$ with $m \geq 0$, then using the same approach one can obtain that for each homogeneous differential operator D^γ with $|\gamma| \leq m$ the vector $V_\phi^{(\gamma)} := \left(\left(D^\gamma \phi \right)(\alpha) \right)_{\alpha\in\Lambda}$ is a left eigenvector of the matrix

$$A^{(\gamma)} := \left(\sum_{m=1}^{M-1} 2^{m|\gamma|} p_{m,2^m \alpha - \beta} \right)_{\alpha,\beta\in\Lambda}$$

with eigenvalue 1.

4.2 Multi-scale subdivision and matrix subdivision

We are now ready to see how multi-scale subdivision is a special case of matrix subdivision. We follow [CDL] for basic results on matrix subdivision. Assume that $\mathcal{S}$ is a UQC $M-$scale scheme given by masks $\{P_m\}_{m=1,\dots,M-1}$. By Theorem 4.9 the scheme has an $\mathcal{S}$-refinable function $\phi \in C(\mathbb{R}^d)$ with an $M-$scale relation (4.17). We now go back to the construction (3.11) and define

$$\Sigma = \bigcup_{m=0}^{M-2} \Phi_m, \qquad \Phi_m := \left\{ \phi(2^m \cdot -k) \;\middle|\; k \in \{0,\dots,2^m-1\}^d \right\}. \tag{4.26}$$

By virtue of (3.12), the FSI space $S(\Sigma)$ is two-scale refinable and thus there exists matrices $A_k \in M_{|\Sigma|\times|\Sigma|}(\mathbb{R})$, $k \in \mathbb{Z}^d$ so that

$$\Sigma' = \sum_{k\in\mathbb{Z}^d} A_k \Sigma(2\cdot -k)'. \tag{4.27}$$

The corresponding $|\Sigma|-th$ dimensional matrix subdivision process is defined as follows. We use the matrices $\{A_k\}$ of (4.27) as the masks of the subdivision algorithm. For any given initial sequence of data vectors $\vec{f}_k^0 := \left(f_k^{0,1},\dots,f_k^{0,|\Sigma|}\right)$ we iterate

$$\vec{f}_\alpha^j = \sum_{k\in\mathbb{Z}^d} \vec{f}_k^{j-1} A_{\alpha-2k}.$$

Under certain conditions on the matrices $\{A_k\}$, the limit of the matrix subdivision process is

$$\vec{f}(x) = \sum_{k\in\mathbb{Z}^d} \vec{f}_k^0 \Sigma(x-k).$$

Example 4.10 Let $\mathcal{S}$ be a univariate three-scale UQC scheme with an $\mathcal{S}$-refinable function $\phi \in C(\mathbb{R})$. Let $P_1 = \{p_{1,k}\}_{k\in\mathbb{Z}}$, $P_2 = \{p_{2,k}\}_{k\in\mathbb{Z}}$ be the two masks of $\mathcal{S}$. In this case $\Sigma = (\phi, \phi(2\cdot), \phi(2\cdot -1))$ and we have the following two-scale relation

$$(\phi, \phi(2\cdot), \phi(2\cdot -1))' = \sum_{k\in\mathbb{Z}} A_k \left(\phi(2\cdot -k), \phi(4\cdot -2k), \phi(4\cdot -(2k+1))\right)',$$

where

$$A_k = \begin{cases} \begin{bmatrix} p_{1,0} & p_{2,0} & p_{2,1} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & k=0, \\ \begin{bmatrix} p_{1,1} & p_{2,2} & p_{2,3} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & k=1, \\ \begin{bmatrix} p_{1,k} & p_{2,2k} & p_{2,2k+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & else. \end{cases} \tag{4.28}$$

♦

As one can see, in some sense the matrix representation of multi-scale subdivision is somewhat redundant. To strengthen this observation let us see the equivalence of the necessary conditions for uniform convergence.

Example 4.11 Let $\mathcal{S}$ be a univariate three-scale scheme with masks $P_1 = \{p_{1,k}\}_{k\in\mathbf{Z}}$, $P_2 = \{p_{2,k}\}_{k\in\mathbf{Z}}$. First assume that the masks of $\mathcal{S}$ satisfy the necessary conditions (4.14). As we have seen, the representation of $\mathcal{S}$ as a matrix subdivision process, denoted by $\mathcal{S}_M$, is defined by the matrices $\{A_k\}$ of the form (4.28). By Proposition 2.2 in [CDL] a necessary condition for $\mathcal{S}_M$ to converge uniformly is that the matrices

$$B_0 := \sum_{k\in\mathbf{Z}} A_{2k}^t, \quad B_1 := \sum_{k\in\mathbf{Z}} A_{2k+1}^t,$$

have a joint eigenvector corresponding to the eigenvalue 1. In our case the matrices B_0, B_1 can be easily computed using (4.14) and (4.28)

$$B_0 = \sum_{k\in\mathbf{Z}} A_{2k}^t = \begin{pmatrix} \sum_{k\in\mathbf{Z}} p_{1,2k} & 1 & 0 \\ \sum_{k\in\mathbf{Z}} p_{2,4k} & 0 & 0 \\ \sum_{k\in\mathbf{Z}} p_{2,4k+1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 1 & 0 \\ C_2 & 0 & 0 \\ C_2 & 0 & 0 \end{pmatrix},$$

$$B_1 = \sum_{k \in \mathbb{Z}} A'_{2k+1} = \begin{pmatrix} \sum_{k\in\mathbb{Z}} p_{1,2k+1} & 0 & 1 \\ \sum_{k\in\mathbb{Z}} p_{2,4k+2} & 0 & 0 \\ \sum_{k\in\mathbb{Z}} p_{2,4k+3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 & 1 \\ C_2 & 0 & 0 \\ C_2 & 0 & 0 \end{pmatrix}.$$

Since $C_1 + C_2 = 1$, it is easy to see that $(1, C_2, C_2)$ is a joint eigenvector for B_0, B_1 corresponding to the eigenvalue 1. The opposite is also true. Assume that $x = (x_1, x_2, x_3)$ is a joint eigenvector for B_0, B_1 corresponding to the eigenvalue 1. Since B_0, B_1 have the form

$$B_0 = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} \beta_1 & 0 & 1 \\ \beta_2 & 0 & 0 \\ \beta_3 & 0 & 0 \end{pmatrix},$$

we can apply B_0, B_1 on x and verify that $\alpha_1 = \beta_1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3$ and $\alpha_1 + \alpha_2 = 1$. These are exactly the necessary conditions for a UC three-scale scheme.

◆

In the analysis of matrix subdivision the notion of stability is important ([CDL], [CDP], [RS2]). We now show that the set Σ of (4.27) is not an L_∞-stable basis for $S(\Sigma)$ in the case of multi-scale subdivision with more than two scales.

Theorem 4.12 Assume $\mathcal{S}$ is a UQC M-scale scheme, $M > 2$, for which conditions (4.14) hold. Let $\phi \in C(\mathbb{R}^d)$ be the corresponding $\mathcal{S}$-refinable function. Then the set Σ of (4.26) is not an L_∞-stable basis for $S(\Sigma)$.

Proof Assume that the set Σ is L_∞-stable. This means that there exists a constant $A > 0$ such that for any $\{g_{m,k}\}$, $m = 0, \dots, M-2$ we have that

$$A \left\| \{g_{n,k}\} \right\|_{l_\infty(\mathbb{Z}^d)} \le \left\| \sum_{m=0}^{M-2} \sum_{k \in \mathbb{Z}^d} g_{m,k} \phi(2^m x - k) \right\|_{L_\infty(\mathbb{R}^d)}, \qquad 0 \le n \le M-2. \tag{4.29}$$

By Theorem 4.9 the following partition of unity property is valid

$$\sum_{k \in \mathbb{Z}^d} \phi(\cdot - k) = 1.$$

Let $\mathrm{B}_j := \left\{\beta_{j,m}\right\}_{m=0}^{M-2}$, $\beta_{j,m} \in \mathbb{R}$, be any sequence such that $\beta_{j,0} \underset{j\to\infty}{\to} +\infty$ and $\sum_{m=0}^{M-2} \beta_{j,m} = 1$. The partition of unity and the compact support properties of ϕ imply that

$$\sum_{m=0}^{M-2} \beta_{j,m} \sum_{k\in\mathbf{Z}^d} \phi\left(2^m \cdot - k\right) = 1, \qquad \forall j .$$

This leads to a contradiction since by (4.29)

$$1 = \left\| \sum_{m=0}^{M-2} \beta_{j,m} \sum_{k\in\mathbf{Z}^d} \phi\left(2^m \cdot - k\right) \right\|_{L_\infty\left(\mathbb{R}^d\right)} \geq A\beta_{j,0} \underset{j\to\infty}{\to} +\infty .$$

Therefore the set Σ is not L_∞-stable.

◆

Remark Using a similar approach, it is easy to see that Σ is also not L_p-stable for all $1 \leq p < \infty$.

4.3 Analysis of convergence and smoothness, the univariate case

As already pointed out, in some cases, the analysis of multi-scale subdivision can be carried out using the two-scale matrix subdivision formulation. However, this is not true for the case of quasi convergence. In this section we assume the dimension $d = 1$ and present a simple and direct approach to the analysis of multi-scale subdivision schemes that follows Section 2.3 in [Dy]. We frequently make use of z-transforms. For any given data $f = \{f_k\}_{k\in\mathbf{Z}}$ and mask $P_m := \{p_{m,k}\}_{k\in\mathbf{Z}}$, the convolution $P_m * f$ given by $(P_m * f)_\alpha = \sum_{k\in\mathbf{Z}} p_{m,\alpha-2^m k} f_k$ can be represented in z-transform notation by $(P_m * f)(z) = P_m(z) f\left(z^{2^m}\right)$ where $P_m(z) = \sum_{k\in\mathbf{Z}} p_{m,k} z^k$ and $f(z) = \sum_{k\in\mathbf{Z}} f_k z^k$.

Lemma 4.13 If the masks $\{P_m\}$ satisfy conditions (4.14), then for $m = 1, \ldots, M-1$

$$\prod_{n=0}^{m-1}\left(1+z^{2^n}\right) \Big| P_m(z).$$

Proof Fix $1 \le m \le M-1$. Since P_m fulfils (4.14), we have

$$\sum_{k\in Z} p_{m,2^m k+\gamma} = C_m, \qquad \gamma \in E_m^1.$$

Whenever the mask P_m is of finite support, $P_m(z)$ is a polynomial. Thus, we can change summation and rewrite $P_m(z)$ as

$$P_m(z) = \sum_{\gamma=0}^{2^m-1} \sum_{k\in Z} p_{m,2^m k+\gamma} z^{2^m k+\gamma} = \sum_{\gamma=0}^{2^m-1} z^\gamma \sum_{k\in Z} p_{m,2^m k+\gamma} z^{2^m k}.$$

Assume $z \ne 1$ with $z^{2^m} = 1$. Then

$$P_m(z) = C_m \sum_{\gamma=0}^{2^m-1} z^\gamma = C_m \frac{1-z^{2^m}}{1-z} = 0.$$

Since $P_m(z) = 0$ for any $z \ne 1$ which is a 2^m – unit root, we have

$$\frac{1-z^{2^m}}{1-z} = \prod_{n=0}^{m-1}\left(1+z^{2^n}\right) \Big| P_m(z).$$

♦

Theorem 4.14 Let $\mathbb{S}$ be an $M-$scale subdivision scheme with masks $\{P_m\}$ satisfying conditions (4.14). Then there exists an $M-$scale subdivision scheme $\mathbb{DDS}_1$ which generates the first order divided differences

$$\left(df^{j+1,M-1},\ldots,df^{j+M-1,1}\right)=\mathbb{DDS}_1\left(df^{j,M-1},\ldots,df^{j+M-2,1}\right),$$

where

$$\left(df^{j,m}\right)_k:=2^j\Delta f_k^{j,m}, \qquad \Delta f_k^{j,m}:=\left(f_{k+1}^{j,m}-f_k^{j,m}\right).$$

Proof We need only prove for $M\geq 3$, since the case $M=2$ is treated in [Dy, Proposition 3.1]. Let $f^{j,m}(z)=\sum_{k\in\mathbf{Z}} f_k^{j,m}z^k$. Then the divided difference sequence $h^{j,m}:=df^{j,m}$ formally satisfies

$$h^{j,m}(z):=2^j\frac{1-z}{z}f^{j,m}(z), \qquad m=1,\ldots,M-1.$$

Therefore

$$f^{j,m}(z)=2^{-j}\frac{z}{1-z}h^{j,m}(z).$$

Since $\{P_m\}$ satisfy (4.14), by Lemma 4.13 for each $m=1,\ldots,M-1$ the following mask is a polynomial whenever P_m is

$$Q_m(z):=\frac{2^m z^{2^m-1}}{\prod_{n=0}^{m-1}\left(1+z^{2^n}\right)}P_m(z). \tag{4.30}$$

As we assumed $M\geq 3$ there are two cases. For $1\leq m\leq M-2$ we obtain

$$\begin{aligned}
h^{j+m,M-m-1}(z)&=2^{j+m}\frac{1-z}{z}f^{j+m,M-m-1}(z)\\
&=2^{j+m}\frac{1-z}{z}\left(f^{j+m,M-m-2}(z)+P_m(z)f^{j,M-1}\left(z^{2^m}\right)\right)\\
&=2^{j+m}\frac{1-z}{z}\left(2^{-(j+m)}\frac{z}{1-z}h^{j+m,M-m-2}(z)+P_m(z)2^{-j}\frac{z^{2^m}}{1-z^{2^m}}h^{j,M-1}\left(z^{2^m}\right)\right)\\
&=h^{j+m,M-m-2}(z)+Q_m(z)h^{j,M-1}\left(z^{2^m}\right).
\end{aligned}$$

By the same method, for $m=M-1$ we have

$$h^{j+M-1,1}(z)=Q_{M-1}(z)h^{j,M-1}\left(z^{2^{M-1}}\right).$$

♦

Definition 4.15 Let $\mathbb{S}$ be an $M-$scale subdivision scheme with masks satisfying conditions (4.14) and let $\mathbb{DDS}_1$ be the corresponding **divided difference scheme.** We define the $M-$scale

difference scheme $\mathbb{DS}_1$ using the masks $2^{-m}Q_m(z)$, $m=1,\dots,M-1$ where $\{Q_m\}$ are defined by (4.30). The scheme $\mathbb{DS}_1$ generates the differences $\Delta f^{j,m} := \left(f_{k+1}^{j,m} - f_k^{j,m}\right)$.

In the two-scale subdivision the approach of computing the smoothness of a scheme by analyzing the corresponding difference schemes is well established. Here we generalize this approach to the multi-scale case. We require the following simple lemma for our first such result.

Lemma 4.16 Let $1 \neq z \in \mathbb{C}$ be a 2^m-th unit root for some $m \geq 1$. Then $z^{2^n} = -1$ for some $0 \leq n < m$.

Proof We use induction. For $m=1$ the claim is obvious as $z \neq 1$ implies $z=-1$. Assume the claim is true for all $1 \leq m' < m$. Since $z^{2^{m-1}}$ is a unit root of degree two there are two possibilities. If $z^{2^{m-1}} = -1$ we can chose $n = m-1$. Else we must have $z^{2^{m-1}} = 1$ and by induction there exists $0 \leq n < m-1$ such that $z^{2^n} = -1$.

♦

Theorem 4.17 Let $\mathbb{S}$ be a multi-scale scheme for which conditions (4.14) hold. Then,

1. If $\mathbb{S}$ is uniformly convergent then the difference scheme $\mathbb{DS}_1$ converges uniformly to zero for any initial input.
2. If $\mathbb{DS}_1$ converges uniformly to zero for any initial input then $\mathbb{S}$ is uniformly quasi convergent.

Proof To make the proof shorter we assume that $M \geq 3$. The proof of the case $M=2$ is found in [Dy], Theorem 3.2. The proof of the first direction ($1 \to 2$) follows from the triangle inequality. If $\mathbb{S}$ is UC, then for any initial f^0 we have that $\mathbb{S}^\infty f^0 = f = \sum_{m=1}^{M-1} f_m$ with $f_m \in C(\mathbb{R})$. For $k \in \mathbb{Z}$ and $m=1,\dots,M-1$ we have

$$\begin{aligned}\left|\left(\Delta f^{j,m}\right)_k\right| &= \left|f_{k+1}^{j,m} - f_k^{j,m}\right| \\ &\leq \left|f_{k+1}^{j,m} - f_m\left(2^{-j}k+2^{-j}\right)\right| + \left|f_k^{j,m} - f_m\left(2^{-j}k\right)\right| + \left|f_m\left(2^{-j}k+2^{-j}\right) - f_m\left(2^{-j}k\right)\right|.\end{aligned}$$

Therefore the difference scheme converges uniformly to zero.

To see the (weaker) opposite direction we begin by observing that if $\mathbb{DS}_1$ converges uniformly to zero on some initial input, then the partial difference scheme $\widetilde{\mathbb{DS}_1}$ generating only $\Delta f^{j,M-1}$ also converges uniformly to zero on the same input.

Remark Actually, it is also true that convergence to zero of $\widetilde{\mathbb{DS}_1}$ implies convergence to zero of $\mathbb{DS}_1$. This is because

$$\Delta f^{j,m} = \sum_{r=1}^{m}\left(2^{-(M-r)} Q_{M-r}\right) * \Delta f^{j-(M-r),M-1}, \qquad m = 1,\ldots,M-1,$$

where $\{Q_m\}$ are defined by (4.30).

For any initial sequence $f^0 \in l_\infty(\mathbb{Z})$ such that $\|f^0\|_\infty = 1$ we have

$$\left\|\widetilde{\mathbb{DS}}_1{}^j f^0\right\|_\infty = \left\|\sum_k f_k^0 \left(\widetilde{\mathbb{DS}}_1{}^j \delta\right)_{\cdot-2^j k}\right\|_\infty \le \left\|\sum_k \left(\widetilde{\mathbb{DS}}_1{}^j \delta\right)_{\cdot-2^j k}\right\|_\infty \le A\left\|\widetilde{\mathbb{DS}}_1{}^j \delta\right\|_\infty.$$

where the constant A is derived from

$$supp\left(\widetilde{\mathbb{DS}}_1{}^j \delta\right) \subseteq \left(2^j - 1\right)\left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} supp\left(2^{-m} Q_m\right)\right\rangle,$$

and the finite support of the masks $\{2^{-m} Q_m\}$. Since $\widetilde{\mathbb{DS}}_1$ converges uniformly to zero there exists $0 < \mu < 1$ and a scale J_0 such that for all initial data $f^0 \in l_\infty(\mathbb{Z})$

$$\left\|\widetilde{\mathbb{DS}}_1{}^{J_0} f^0\right\|_\infty < \mu \left\|f^0\right\|_\infty. \tag{4.31}$$

We now use (4.4) to compute the difference between two refinement levels

$$\begin{aligned}
&F^{j+1}(x) - F^j(x) \\
&= \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}} f_k^{j+m,M-m} N\left(2^{j+m}x - k\right) - \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}} f_k^{j+m-1,M-m} N\left(2^{j+m-1}x - k\right) \\
&= \sum_{m=1}^{M-2}\sum_{k\in\mathbb{Z}} \left(f_k^{j+m,M-m-1} + \left(P_m * f^{j,M-1}\right)_k\right) N\left(2^{j+m}x - k\right) + \sum_{k\in\mathbb{Z}} \left(P_{M-1} * f^{j,M-1}\right)_k N\left(2^{j+M-1}x - k\right) \\
&\qquad - \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}} f_k^{j+m-1,M-m} N\left(2^{j+m-1}x - k\right) \\
&= \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}} \left(P_m * f^{j,M-1}\right)_k N_2\left(2^{j+m}x - k\right) - \sum_{k\in\mathbb{Z}} f_k^{j,M-1} N\left(2^j x - k\right).
\end{aligned}$$

Denote by $U = \{1/2, 1, 1/2\}$ the (two-scale) mask of N_2. As we assumed $M \ge 3$, it is possible with a slight abuse of the convolution notation, to write the difference between levels as

$$F^{j+1}(x) - F^j(x) = \sum_{k\in\mathbb{Z}} \left(\left(\sum_{m=1}^{M-1} U^{M-m-1} * P_m - U^{M-1}\right) * f^{j,M-1}\right)_k N_2\left(2^{j+M-1}x - k\right).$$

We see that the control points of the polygon $F^{j+1}(x)-F^{j}(x)$ are the coefficients of $D(z)f^{j,M-1}\left(z^{2^{M-1}}\right)$ where

$$D(z):=\sum_{m=1}^{M-1}\left(\prod_{r=1}^{M-m-1}U\left(z^{2^{r-1}}\right)\right)P_m\left(z^{2^{M-m-1}}\right)-\prod_{m=1}^{M-1}U\left(z^{2^{m-1}}\right). \tag{4.32}$$

Next we show that $D(z)$ has all the unit roots of degree 2^{M-1}. First observe that the mask U (as a two-scale scheme) and the masks $\{P_m\}$ fulfill the necessary conditions (4.14). This implies that $U(1)=2$, $U(-1)=0$. We begin with the case $z=1$. Using (4.32) we have,

$$D(1)=\sum_{m=1}^{M-1}2^{M-m-1}2^{m}C_m-2^{M-1}=2^{M-1}\sum_{m=1}^{M-1}C_m-2^{M-1}=0.$$

Now assume that $z\neq 1$ is a unit root of degree 2^{M-1}. First, we analyze the product $\prod_{m=1}^{M-1}U\left(z^{2^{m-1}}\right)$ appearing in (4.32). By Lemma 4.16 for some $1\leq m\leq M-1$ we must have $z^{2^{m-1}}=-1$. Since $U(-1)=0$, this product is zero for any choice of unit root $\neq 1$. We now prove that the sum appearing in (4.32) is also zero. We analyze separately each term $\left(\prod_{r=1}^{M-m-1}U\left(z^{2^{r-1}}\right)\right)P_m\left(z^{2^{M-m-1}}\right)$ for $1\leq m\leq M-1$. Observe that since z is a unit root of degree 2^{M-1}, $z^{2^{M-m-1}}$ is a unit root of degree 2^{m}. There are two cases: if $z^{2^{M-m-1}}\neq 1$, then by Lemma 4.13 $P_m\left(z^{2^{M-m-1}}\right)=0$. Else $z^{2^{M-m-1}}=1$. In such a case, using again Lemma 4.16, we must have $\left(\prod_{r=1}^{M-m-1}U\left(z^{2^{r-1}}\right)\right)=0$. Combining the last two arguments we conclude that $D(z)$ is also zero for any unit root $z\neq 1$ of degree 2^{M-1}. Thus, $D(z)$ has all the unit roots of degree 2^{M-1} and can be factored to

$$D(z)=\frac{1-z^{2^{M-1}}}{z^{2^{M-1}}}E(z),$$

where $E(z)=\sum_{k\in Z}e_k z^k$ is a polynomial (finite support) whenever $\{P_m\}$ are. Therefore

$$D(z)f^{j,M-1}\left(z^{2^{M-1}}\right)=E(z)\frac{1-z^{2^{M-1}}}{z^{2^{M-1}}}f^{j,M-1}\left(z^{2^{M-1}}\right)=E(z)\sum_{k\in \mathbf{Z}}\left(f_{k+1}^{j,M-1}-f_{k}^{j,M-1}\right)z^{2^{M-1}k}. \tag{4.33}$$

We can now conclude that $\{F^j(x)\}$ is a Cauchy sequence. Using (4.31) and (4.33) we have the estimate

$$\begin{aligned}\left|F^{j+1}(x)-F^j(x)\right| &= \left\|D(z)f^{j,M-1}\left(z^{2^{M-1}}\right)\right\|_\infty \\ &\le C_E\left\|\widetilde{\mathbb{DS}}_1^{\,j} f^0\right\|_\infty \\ &\le C_E \max_{0\le r\le J_0}\left\|\widetilde{\mathbb{DS}}_1^{\,r} f^0\right\|_\infty \mu^{\left[\frac{j}{J_0}\right]},\end{aligned}$$

with $C_E := \max_{\gamma\in E^1_{M-1}}\left\{\sum_{k\in\mathbb{Z}}\left|e_{\gamma-2^{M-1}k}\right|\right\}$.

Since $\{F^j(x)\}$ are continuous and converge uniformly, the limit $f(x)$ is also continuous. Consequently, $\mathbb{S}$ is UQC.

♦

Example 4.18 To see that the convergence to zero of the difference scheme $\mathbb{DS}_1$ does not imply the uniform convergence of $\mathbb{S}$ but the weaker uniform quasi convergence, we give the following counter-example. Let $\alpha\in\mathbb{R}$ and let $\mathbb{S}$ be the three-scale scheme defined by the centered masks

$$P_1=\alpha\left\{\frac{1}{2},1,\frac{1}{2}\right\},\qquad P_2=(1-\alpha)\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2},\frac{1}{4}\right\}.$$

Observe that $\mathbb{S}$ is interpolatory and that $P_1(z)=\alpha U(z)$, $P_2(z)=(1-\alpha)U(z)U(z^2)$ where $U(z)$ is the mask corresponding to N_2. As we shall now see, this means that $\mathbb{S}$ is actually a three-scale representation of a two-scale scheme. We argue that for the initial data $f^0=\delta$ the following holds:

$$\begin{aligned} f^{j,2}(x) &:= \sum_{k\in\mathbb{Z}} f_k^{j,2}N_2\left(2^j x-k\right)=K_j(\alpha)N_2(x), \\ f^{j+1,1}(x) &:= \sum_{k\in\mathbb{Z}} f_k^{j+1,1}N_2\left(2^{j+1}x-k\right)=\left(1-K_j(\alpha)\right)N_2(x),\end{aligned}\tag{4.34}$$

where

$$K_j(\alpha)=\frac{(\alpha-1)^{j+1}-1}{\alpha-2},\qquad j\ge 0.$$

If (4.34) holds, then $F^j(x)=f^{j,2}(x)+f^{j+1,1}(x)=N_2(x)$ for $j\ge 0$. This implies by linearity that $\mathbb{S}$ is UQC. We now verify our claim using induction. It is easy to see that for the initial data $f^{0,2}=\delta, f^{1,1}=0$ (4.34) is valid since $K_0(\alpha)=1$. Assume that (4.34) holds for j. By virtue of

$$f^{j+1,2}=f^{j+1,1}+P_1 * f^{j,2},$$

and the properties of the mask P_1 we have

$$\begin{aligned} f^{j+1,2}(x) &= f^{j+1,1}(x)+\alpha f^{j,2}(x) \\ &= \left(1-K_j(\alpha)\right)N_2(x)+\alpha K_j(\alpha)N_2(x) \\ &= \left(1+(\alpha-1)K_j(\alpha)\right)N_2(x) \\ &= \left(1+(\alpha-1)\frac{(\alpha-1)^{j+1}-1}{\alpha-2}\right)N_2(x) \\ &= K_{j+1}(\alpha)N_2(x). \end{aligned}$$

Using the properties of the mask P_2 we also obtain

$$\begin{aligned} f^{j+2,1}(x) &= (1-\alpha)f^{j,2}(x) \\ &= (1-\alpha)K_j(\alpha)N_2(x) \\ &= (1-\alpha)\frac{(\alpha-1)^{j+1}-1}{\alpha-2}N_2(x) \\ &= \left(-K_{j+1}(\alpha)-\frac{2-\alpha}{\alpha-2}\right)N_2(x) \\ &= \left(1-K_{j+1}(\alpha)\right)N_2(x). \end{aligned}$$

Thus, (4.34) holds and $\mathbb{S}$ is UQC. It is easy to see that for the choice $0<\alpha<2$ the scheme is UC. Let us now choose $2<\alpha<3$. For such a choice, by virtue of (4.34), the components $f^{j,2}, f^{j+1,1}$ diverge and the scheme is not UC. On the other hand for $2<\alpha<3$ the difference scheme $\mathbb{DS}_1$ uniformly converges to zero. This is because

$$\left|\Delta f_k^{j,2}\right| \le 2^{-j}K_j(\alpha) \le C\left(\frac{\alpha-1}{2}\right)^j \underset{j\to\infty}{\to} 0,$$

$$\left|\Delta f_k^{j+1,1}\right| \le 2^{-(j+1)}\left|1-K_j(\alpha)\right| \underset{j\to\infty}{\to} 0.$$

We now proceed with analysis of smoothness. First we require the following lemma.

Lemma 4.19 Let $\{g_j\}_{j\ge1}$ be a sequence of piecewise constant univariate functions supported on some finite interval $[a,b]$ such that

$$g_j\Big|_{\left[2^{-(j+n)}k,\,2^{-(j+n)}(k+1)\right)} = \alpha_{j,k}, \qquad j\ge1, \quad k\in\mathbb{Z},$$

for some $n \ge 0$. Assume that g_j converge uniformly to $g \in C([a,b])$ on the dyadic points of $[a,b]$ in the following sense

$$\left\| g_j\left(2^{-(j+n)}k\right) - g\left(2^{-(j+n)}k\right) \right\|_\infty \to 0.$$

Then g_j converge uniformly to g in $[a,b]$.

Proof By shifting the index of the elements in the sequence $\{g_j\}$ we can assume that $n = 0$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|2^{-j}k - x| < \delta$ implies $|g(2^{-j}k) - g(x)| < \frac{\varepsilon}{2}$ for all $x \in [a,b]$. Also using the uniform convergence on dyadic points there exists $J(\delta,\varepsilon)$ such that for any $x \in [a,b]$ and $j > J(\delta,\varepsilon)$ we can choose a dyadic point $2^{-j}k_j$ such that

1. $\left|g_j(2^{-j}k_j) - g(2^{-j}k_j)\right| < \frac{\varepsilon}{2}$,
2. $g_j(x) = g_j(2^{-j}k_j)$,
3. $|2^{-j}k_j - x| < \delta$.

Consequently

$$\begin{aligned}|g_j(x) - g(x)| &= \left|g_j(2^{-j}k_j) - g(x)\right| \\ &\le \left|g_j(2^{-j}k_j) - g(2^{-j}k_j)\right| + \left|g(2^{-j}k_j) - g(x)\right| < \varepsilon.\end{aligned}$$

♦

The following result is a generalization of Theorem 3.4 in [Dy]. It states sufficient conditions under which a scheme is convergent and smooth.

Theorem 4.20 Let $r \in \mathbb{N}$ and let $\mathcal{S}$ be a univariate M – scale subdivision scheme with

$$P_m(z) = \left(\frac{\prod_{n=0}^{m-1}\left(1 + z^{2^n}\right)}{2^m z^{2^m - 1}} \right)^r Q_m(z), \qquad m = 1,\dots,M-1, \tag{4.35}$$

such that the multi-scale scheme defined by $\{Q_m\}$ is UQC. Then,

1. $\phi := \mathcal{S}^\infty \delta \in C^r(\mathbb{R})$,

(4.36)

2. $\frac{d^\gamma}{dx^\gamma}\phi = \mathbb{DDS}_\gamma^\infty\left(\Delta^\gamma f^0\right)$, $1 \le \gamma \le r$, where $\mathbb{DDS}_\gamma$ generates the γ-order divided differences

$$2^{j\gamma}\left(\Delta^{\gamma} f^{j,m}\right)_{\alpha} := 2^{j\gamma}\sum_{k=0}^{\gamma}\binom{\gamma}{k}(-1)^{\gamma-k} f^{j,m}_{\alpha+k}.$$

3. The $\mathbb{S}$-refinable function ϕ provides approximation order r.

Proof We first treat the case $r=1$. By Theorem 4.14, the scheme corresponding to $\{Q_m\}$ is exactly the divided difference scheme $\mathbb{DDS}_1$ corresponding to $\mathbb{S}$. Recall that $\mathbb{DDS}_1$ generates for any initial data f^0 the divided difference data $df^{j,m} := \left\{2^j \Delta f_k^{j,m}\right\}_{k\in\mathbf{Z}}$, $j \geq 0$, $m=1,\dots,M-1$. We now construct for $f^0=\delta$ the sequence of functions

$$g_j(x) := \sum_{m=1}^{M-1} 2^{j+m-1} \sum_{k\in\mathbf{Z}} \Delta f_k^{j+m-1,M-m} \chi_{\left[2^{-(j+m-1)}k,\,2^{-(j+m-1)}(k+1)\right)}(x).$$

Using Lemma 4.19, the uniform quasi convergence of $\mathbb{DDS}_1$ implies that the above sequence converges uniformly to the limit function $g=\mathbb{DDS}_1^{\infty}(\Delta\delta)\in C(\mathbb{R})$. Since all functions considered here are of joint compact support, we conclude that

$$\int_{-\infty}^{x} g_j(t)\,dt \underset{j\to\infty}{\rightarrow} \int_{-\infty}^{x} g(t)\,dt.$$

Let us denote $\phi(x) := \int_{-\infty}^{x} g(t)\,dt$. By definition of $g_j(x)$

$$\int_{-\infty}^{x} g_j(t)\,dt = \sum_{m=1}^{M-1}\sum_{k\in\mathbf{Z}} f_k^{j+m-1,M-m} N_2\left(2^{j+m-1}x-k\right) = F^j(x) \underset{j\to\infty}{\rightarrow} \phi(x).$$

We conclude that $\mathbb{S}^{\infty}\delta = \phi \in C^1(\mathbb{R})$ and that $\phi' = \mathbb{DDS}_1^{\infty}\left(\Delta f^0\right)$. By repeated application of the above argument we show that (4.36) holds. The approximation order of ϕ is a consequence of Corollary 3.9, since ϕ can be represented as a convolution of a continuous function and a B-spline of order r.

♦

Finally, we present an application of the analysis tools of this section.

Example 4.21 A scheme $\mathbb{S} \in \{\mathbb{S}_{3,\beta}\}$, where $\{\mathbb{S}_{3,\beta}\}$ is the parametric family introduced in Example 4.6, defined by the masks

$$P_{\beta,1}(z) = \frac{\beta}{2} z^{-1}(1+z)^2, \qquad P_{\beta,2}(z) = \frac{1-\beta}{16} z^{-4}(1+z)^4\left(1+z^2\right)^2,$$

is a C^1 scheme for the choice $-1/3 < \beta < 1$.

Proof By Theorem 4.20 it is sufficient to show that the three-scale scheme $\mathcal{S}$ defined by the masks

$$Q_{\beta,1}(z) = \beta(1+z), \qquad Q_{\beta,2}(z) = \frac{1-\beta}{4} z^{-1}(1+z)^3(1+z^2),$$

is UQC. By Theorem 4.17 this is true if the partial difference scheme $\widetilde{\mathcal{DS}}_1$ converges uniformly to zero. Thus, we need to prove that the scheme given by the masks

$$R_{\beta,1}(z) = \beta z, \qquad R_{\beta,2}(z) = \frac{1-\beta}{4} z^2(1+z)^2,$$

contracts any initial data to zero. The scheme $\widetilde{\mathcal{DS}}_1$ is defined by

$$f_k^{j+2,2} = \left(R_{\beta,1} * f^{j+1,2}\right)_k + \left(R_{\beta,2} * f^{j,2}\right)_k = \begin{cases} \beta f^{j+1,2}_{\frac{k-1}{2}} & k(\mathrm{mod}\, 2) \equiv 1 \\ 0 & else \end{cases} + \begin{cases} \frac{1-\beta}{4} f^{j,2}_{\frac{k-2}{4}} & k \equiv 2(\mathrm{mod}\, 4), \\ \frac{1-\beta}{2} f^{j,2}_{\frac{k-3}{4}} & k \equiv 3(\mathrm{mod}\, 4), \\ \frac{1-\beta}{4} f^{j,2}_{\frac{k-4}{4}} & k \equiv 0(\mathrm{mod}\, 4), \\ 0 & else. \end{cases}$$

Observe that the difference scheme "contracts" whenever

$$\max\left(|\beta| + \left|\frac{1-\beta}{2}\right|, \left|\frac{1-\beta}{4}\right|\right) < 1.$$

We see that for the range $-1/3 < \beta < 1$, we obtain a contraction, implying that the scheme $\mathcal{S}_{3,\beta}$ is C^1. Observe that as $\beta \to 1$ the scheme converges to the interpolatory linear B-spline scheme. On the other hand the solution to

$$\min_{-1<\beta<1} \max\left(|\beta| + \left|\frac{1-\beta}{4}\right|, \left|\frac{1-\beta}{2}\right|\right),$$

is at $\beta = 0$. This corresponds to maximal possible Hölder exponent of the first derivative. We see a tradeoff between "near interpolation" for values of β just below 1 and higher smoothness at zero. Note that for $\beta = 0$ we obtain that

$$P_{\beta,1}(z)=0\,, \quad P_{\beta,2}(z)=\frac{1}{16}z^{-4}(1+z)^4(1+z^2)^2\,.$$

Thus, the corresponding generating function has a (non-binary) two-scale relation

$$\phi(x)=\sum_{k\in\mathbb{Z}} p_{2,k}\phi(4x-k)\,.$$

5 Non-stationary wavelets

We now describe non-stationary multiresolution constructions that can be used to decompose an initial non-refinable SI space into a sum of wavelet subspaces. Unlike the construction (3.11), the proposed constructions are stable and do not create redundant wavelet subspaces. The decompositions described below will become meaningful in Sections 6.4 and 6.5, where the inheritance of approximation properties from the initial SI space is discussed in detail.

Our first results are simple modifications of the classical "symbol approach" [Ch] to wavelet construction which previously assumed two-scale refinability of the scaling function. Assume $\rho,\psi \subset S(\varphi)^{1/2}$ where $\varphi \in L_2(\mathbb{R})$ is stable. In refinable setting one has the special case $\rho=\varphi$ so that $S(\varphi)\subset S(\varphi)^{1/2}$. We would like to characterize the cases where $\{\rho,\psi\}$ is a basis for $S(\varphi)^{1/2}$. Define the formal symbols

$$P(w):=\frac{1}{2}\sum_{k\in\mathbb{Z}}p_k e^{-ikw}\text{, where } \rho=\sum_{k\in\mathbb{Z}}p_k\varphi(2\cdot-k).$$
$$Q(w):=\frac{1}{2}\sum_{k\in\mathbb{Z}}q_k w^{-ikw}\text{, where } \psi=\sum_{k\in\mathbb{Z}}q_k\varphi(2\cdot-k). \tag{5.1}$$

We see that the symbols P,Q define the two-scale relations of ρ,ψ. We require that these symbols be taken from the Wiener algebra. This justifies the pointwise validity of (5.1) and later on resolves technical difficulties concerning convergence.

Definition 5.1 Let $f(w)\sim\sum_{k\in\mathbb{Z}^d}f_k e^{-ikw}$ be the Fourier series of $f\in L_2(\mathbb{T}^d)$. Then $f\in\mathbb{W}$, the **Wiener Algebra,** if $\{f_k\}\in l_1(\mathbb{Z}^d)$. Observe that $\mathbb{W}\subset C(\mathbb{T}^d)$.

In this chapter the following partitioning of the lattice $\mathbb{Z}^d$ is useful

$$\mathbb{Z}^d=\bigcup_{e\in E_d}\left(e+2\mathbb{Z}^d\right),\quad E_d:=\{0,1\}^d. \tag{5.2}$$

We begin with a "stability" lemma.

Lemma 5.2 Let $\psi\in S(\varphi)^{1/2}$ have a two-scale relation $Q\in\mathbb{W}$ such that

$$\sum_{e\in E_d}\left|Q(w+\pi e)\right|^2>0,\qquad \forall w\in\mathbb{T}^d,$$

where $\varphi \in L_2(\mathbb{R}^d)$ is stable. Then ψ is a stable generator for $S(\psi)$. In particular $S(\psi)$ is a regular PSI space.

Proof Using the two-scale relation of ψ we have

$$
\begin{aligned}
[\hat{\psi},\hat{\psi}](w) &= \sum_{k\in\mathbb{Z}^d}\left|Q\left(\frac{w}{2}+\pi k\right)\right|^2\left|\hat{\varphi}\left(\frac{w}{2}+\pi k\right)\right|^2 \\
&= \sum_{e\in E_d}\sum_{k\in\mathbb{Z}^d}\left|Q\left(\frac{w}{2}+\pi e+2\pi k\right)\right|^2\left|\hat{\varphi}\left(\frac{w}{2}+\pi e+2\pi k\right)\right|^2 \\
&= \sum_{e\in E_d}\left|Q\left(\frac{w}{2}+\pi e\right)\right|^2\left[\hat{\varphi}\left(\frac{w}{2}+\pi e\right),\hat{\varphi}\left(\frac{w}{2}+\pi e\right)\right].
\end{aligned}
$$

By Theorem 2.20, the stability of φ implies that there exist $0<A\le B<\infty$ such that $A\le[\hat{\varphi},\hat{\varphi}]\le B$ a.e. Thus, we can bound the auto-correlation of ψ by

$$
A\sum_{e\in E_d}\left|Q\left(\frac{w}{2}+\pi e\right)\right|^2 \le [\hat{\psi},\hat{\psi}](w) \le B\sum_{e\in E_d}\left|Q\left(\frac{w}{2}+\pi e\right)\right|^2, \qquad \text{a.e.}
$$

Since $Q\in C(\mathbb{T}^d)$ we have that $B' := 2^d\|Q\|_{C(\mathbb{T}^d)}<\infty$. Also by continuity

$$
A' := \min_{w\in\mathbb{T}^d}\left(\sum_{e\in E_d}\left|Q(w+\pi e)\right|^2\right) > 0.
$$

Hence it follows that

$$
0 < AA' \le [\hat{\psi},\hat{\psi}](w) \le BB' < \infty, \qquad \text{a.e.}
$$

Applying Theorem 2.20 again, we conclude that ψ is a stable generator for its PSI space.

♦

Next we observe that the following result proved in [Ch] for the refinable case $\rho=\varphi$, is still valid for the more general case. Recall (Definition 2.17) that a set of generators Φ for a shift invariant space $S(\Phi)$ is a basis if each $f\in S(\Phi)$ has a unique representation $\hat{f}=\sum_{\phi\in\Phi}\tau_\phi\hat{\phi}$. Note that a basis need not be a stable basis.

Theorem 5.3 Let $\varphi \in L_2(\mathbb{R})$ be a basis for $S(\varphi)$ and let $\rho, \psi \in S(\varphi)^{1/2}$. Assume $P, Q \in \mathbb{W}$ where P, Q are the symbols (5.1). A necessary and sufficient condition for $\{\rho, \psi\}$ to be a basis for $S(\varphi)^{1/2}$ is

$$\Delta_{P,Q}(w) := P(w)Q(w+\pi) - P(w+\pi)Q(w) \neq 0, \qquad \forall w \in \mathbb{T}. \tag{5.3}$$

Furthermore, if φ is stable, then both ρ, ψ are stable.

Proof The proof is similar to the refinable case (see [Ch] Theorem 5.16). We prove only one direction and assume (5.3) holds. Let us define the following matrix

$$M_{P,Q}(w) := \begin{bmatrix} P(w) & Q(w) \\ P(w+\pi) & Q(w+\pi) \end{bmatrix}. \tag{5.4}$$

Observe that $\Delta_{P,Q}(w) = \det M_{P,Q}(w)$. Since $P, Q \in \mathbb{W}$ implies $P(w+\pi), Q(w+\pi) \in \mathbb{W}$ we have, $\Delta_{P,Q} \in \mathbb{W}$. By Wiener's lemma [K] $\Delta_{P,Q} \neq 0$ implies $\Delta_{P,Q}^{-1} \in \mathbb{W}$, so the two functions

$$G(w) := \frac{Q(w+\pi)}{\Delta_{P,Q}(w)} \quad \text{and} \quad H(w) := \frac{-P(w+\pi)}{\Delta_{P,Q}(w)}, \tag{5.5}$$

also belong to the Wiener Algebra. For the matrix $M_{G,H}(w)$ we have the relations

$$M_{P,Q}(w) M_{G,H}^t(w) = M_{G,H}^t(w) M_{P,Q}(w) = I, \qquad \forall w \in \mathbb{T}. \tag{5.6}$$

Also, since $G, H \in \mathbb{W}$

$$G(w) := \frac{1}{2} \sum_{k \in \mathbf{Z}} g_k e^{-ikw}, \; H(w) := \frac{1}{2} \sum_{k \in \mathbf{Z}} h_k e^{-ikw}, \tag{5.7}$$

for some $\{g_k\}, \{h_k\} \in l_1(\mathbb{Z})$. By virtue of (5.6) we have for all $w \in \mathbb{T}$

$$\begin{cases} P(w)\big(G(w)+G(w+\pi)\big) + Q(w)\big(H(w)+H(w+\pi)\big) = 1, \\ P(w)\big(G(w)-G(w+\pi)\big) + Q(w)\big(H(w)-H(w+\pi)\big) = 1, \end{cases}$$

which in view of (5.7) can be written as

$$\begin{cases} P(w) \sum_{k \in \mathbf{Z}} g_{2k} e^{-i2kw} + Q(w) \sum_{k \in \mathbf{Z}} h_{2k} e^{-i2kw} = 1, \\ P(w) \sum_{k \in \mathbf{Z}} g_{2k-1} e^{-i(2k-1)w} + Q(w) \sum_{k \in \mathbf{Z}} h_{2k} e^{-i(2k-1)w} = 1. \end{cases} \tag{5.8}$$

Multiplying the identities (5.8) by $\hat{\varphi}\left(\frac{w}{2}\right)$ and $e^{-iw/2}\hat{\varphi}\left(\frac{w}{2}\right)$, respectively, we obtain

$$\begin{cases} \hat{\varphi}\left(\frac{w}{2}\right) = \sum_k \left(g_{2k} e^{-ikw} P\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right) + h_{2k} e^{-ikw} Q\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right)\right), \\ e^{-iw/2}\hat{\varphi}\left(\frac{w}{2}\right) = \sum_k \left(g_{2k-1} e^{-ikw} P\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right) + h_{2k-1} e^{-ikw} Q\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right)\right). \end{cases}$$

Using (5.1), this is equivalent to

$$\begin{cases} \hat{\varphi}\left(\frac{w}{2}\right) = \sum_k \left(g_{2k} e^{-ikw} \hat{\rho}(w) + h_{2k} e^{-ikw} \hat{\psi}(w)\right), \\ e^{-iw/2}\hat{\varphi}\left(\frac{w}{2}\right) = \sum_k \left(g_{2k-1} e^{-ikw} \hat{\rho}(w) + h_{2k-1} e^{-ikw} \hat{\psi}(w)\right). \end{cases}$$

By taking the inverse Fourier transform of both sides we obtain (at least in the L_2 sense)

$$\begin{cases} 2\varphi(2x) = \sum_k \left(g_{2k} \rho(x-k) + h_{2k} \psi(x-k)\right), \\ 2\varphi(2x-1) = \sum_k \left(g_{2k-1} \rho(x-k) + h_{2k-1} \psi(x-k)\right), \end{cases}$$

which implies

$$\varphi(2x-l) = \frac{1}{2}\sum_k \left(g_{2k-l} \rho(x-k) + h_{2k-l} \psi(x-k)\right), \qquad l \in \mathbb{Z}. \tag{5.9}$$

We have thus shown that $S(\varphi)^{1/2} = S(\rho) + S(\psi)$. Next we wish to see that $\{\rho, \psi\}$ is a basis for $S(\varphi)^{1/2}$. Assume $A(w)\hat{\rho}(w) + B(w)\hat{\psi}(w) \equiv 0$ a.e. for some A, B 2π-periodic functions. By applying the two-scale relations (5.1) we obtain

$$\left(A(2w)P(w) + B(2w)Q(w)\right)\hat{\varphi}(w) \equiv 0.$$

Since we assumed φ is a basis for $S(\varphi)$ we must have a.e.

$$\begin{cases} A(2w)P(w) + B(2w)Q(w) = 0, \\ A(2w)P(w+\pi) + B(2w)Q(w+\pi) = 0, \end{cases}$$

where we have replaced w by $w+\pi$ in the second equality. Since we assumed $M_{P,Q}(w)$ is non-singular for all $w \in \mathbb{T}$ we conclude that $A(w), B(w) \equiv 0$ a.e. and so $\{\rho, \psi\}$ is a basis. Finally, assume φ is stable. We wish to prove both ρ, ψ are stable. But this is an immediate consequence of Lemma 5.2, since $\Delta_{P,Q} \neq 0$ implies that

$$|P(w)|^2 + |P(w+\pi)|^2, |Q(w)|^2 + |Q(w+\pi)|^2 > 0, \qquad \forall w \in \mathbb{T}.$$

◆

Next we discuss a special case of the decomposition $S(\varphi)^{1/2} = S(\rho) + S(\psi)$, by adding an orthogonality constraint $S(\rho) \perp S(\psi)$.

Definition 5.4 Let $\varphi \in L_2(\mathbb{R})$ and $\rho, \psi \in S(\varphi)^{1/2}$. In case $S(\rho) \oplus S(\psi) = S(\varphi)^{1/2}$, we say that the decomposition is **semi-orthogonal** and that ρ, ψ are a **semi-orthogonal** pair.

The term semi-orthogonality comes from the fact that $S(\rho) \perp S(\psi)$ but the shifts of ρ, ψ are not necessarily an orthogonal basis for their span. Assume ρ has a symbol $P \in \mathrm{W}$ so that

$$\hat{\rho}(w) = P\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right).$$

Recall that the natural dual $\tilde{\rho}$ (see Definition 2.25) can be used to compute the orthogonal projection into $S(\rho)$. For the dual we also have the following dual two-scale relation

$$\hat{\tilde{\rho}} = \frac{\hat{\rho}}{[\hat{\rho},\hat{\rho}]} = \frac{1}{[\hat{\rho},\hat{\rho}]} P(2^{-1}\cdot)\hat{\varphi}(2^{-1}\cdot) = \frac{[\hat{\varphi},\hat{\varphi}](2^{-1}\cdot)}{[\hat{\rho},\hat{\rho}]} P(2^{-1}\cdot)\hat{\tilde{\varphi}}(2^{-1}\cdot).$$

So that

$$\hat{\tilde{\rho}} = G^*(2^{-1}\cdot)\hat{\tilde{\varphi}}(2^{-1}\cdot), \qquad G^* := \frac{[\hat{\varphi},\hat{\varphi}]}{[\hat{\rho},\hat{\rho}](2\cdot)} P. \tag{5.10}$$

With the definition

$$G := \overline{G^*}, \tag{5.11}$$

we have the duality relation

$$P(w)G(w) + P(w+\pi)G(w+\pi) \equiv 1. \tag{5.12}$$

Indeed, (5.12) can be obtained from the following calculation

$$P\left(\frac{w}{2}\right)G\left(\frac{w}{2}\right) + P\left(\frac{w}{2}+\pi\right)G\left(\frac{w}{2}+\pi\right)$$
$$= P\left(\frac{w}{2}\right)\frac{[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right)}{[\hat{\rho},\hat{\rho}](w)}\overline{P\left(\frac{w}{2}\right)} + P\left(\frac{w}{2}+\pi\right)\frac{[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right)}{[\hat{\rho},\hat{\rho}](w)}\overline{P\left(\frac{w}{2}+\pi\right)}$$

$$= \frac{1}{[\hat{\rho},\hat{\rho}](w)} \left(\left| P\left(\frac{w}{2}\right) \right|^2 [\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right) + \left| P\left(\frac{w}{2}+\pi\right) \right|^2 [\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right) \right)$$
$$= \frac{[\hat{\rho},\hat{\rho}](w)}{[\hat{\rho},\hat{\rho}](w)} \equiv 1.$$

Equipped with the notion of the dual symbol, we now characterize the univariate semi-orthogonal (wavelet) complements of a given generator in a space of type $S(\varphi)^{1/2}$.

Theorem 5.5 Let $\rho \in S(\varphi)^{1/2}$ with a symbol $P \in \mathbb{W}$, such that φ, ρ are stable. Assume further that $G \in \mathcal{W}$, where G is defined by (5.11). Then, $\psi \in S(\varphi)^{1/2}$ is stable semi-orthogonal complement such that $S(\varphi)^{1/2} = S(\rho) \oplus S(\psi)$ with a symbol $Q \in \mathcal{W}$ if and only if

$$Q(w) = e^{iw} G(w+\pi) K(2w), \qquad \text{where } 0 \neq K \in \mathcal{W}. \tag{5.13}$$

Proof First observe that for any $f \in L_2(\mathbb{T})$, $f \in \mathcal{W}$ implies $f(2\cdot) \in \mathcal{W}$. Also, for any π-periodic function $g \in \mathcal{W}$, we have $g(\cdot/2) \in \mathcal{W}$.

Assume $\psi \in S(\varphi)^{1/2}$ with a symbol of type (5.13). Since $G, K(2\cdot) \in \mathcal{W}$, we also have $Q \in \mathcal{W}$. Also, it is easy to see that the sufficient condition (5.3) is met since using (5.12)

$$\Delta_{P,Q}(w) = e^{i(w+\pi)} P(w) G(w) K(2w) - e^{iw} P(w+\pi) G(w+\pi) K(2w)$$
$$= -e^{iw} K(2w) \neq 0.$$

Thus by Theorem 5.3 we have that $\{\rho,\psi\}$ are a basis for $S(\varphi)^{1/2}$ and that ψ is stable. To see that this is a semi-orthogonal decomposition we compute the inner products of shifts of ρ,ψ for any $j,k \in \mathbb{Z}$,

$$\langle \rho(\cdot - j), \psi(\cdot - k) \rangle$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(j-k)w} \hat{\rho}(w) \overline{\hat{\psi}(w)} dw$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(j-k)w} P\left(\frac{w}{2}\right) \overline{Q\left(\frac{w}{2}\right)} \left| \hat{\varphi}\left(\frac{w}{2}\right) \right|^2 dw$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(j-k)w} \left(P\left(\frac{w}{2}\right) \overline{Q\left(\frac{w}{2}\right)} [\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right) + P\left(\frac{w}{2}+\pi\right) \overline{Q\left(\frac{w}{2}+\pi\right)} [\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right) \right) dw.$$

Using (5.11) and (5.13) we have for any $w \in \mathbb{T}$

$$P\left(\frac{w}{2}\right)\overline{Q\left(\frac{w}{2}\right)}[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right)+P\left(\frac{w}{2}+\pi\right)\overline{Q\left(\frac{w}{2}+\pi\right)}[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right)$$
$$=e^{-iw/2}\overline{K(w)}\left(P\left(\frac{w}{2}\right)G^{*}\left(\frac{w}{2}+\pi\right)[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right)-P\left(\frac{w}{2}+\pi\right)G^{*}\left(\frac{w}{2}\right)[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right)\right)$$
$$=e^{-iw/2}\overline{K(w)}\left(P\left(\frac{w}{2}\right)\frac{[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right)}{[\hat{\rho},\hat{\rho}](w)}P\left(\frac{w}{2}+\pi\right)[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right)-P\left(\frac{w}{2}+\pi\right)\frac{[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}\right)}{[\hat{\rho},\hat{\rho}](w)}P\left(\frac{w}{2}\right)[\hat{\varphi},\hat{\varphi}]\left(\frac{w}{2}+\pi\right)\right)=0.$$

Thus, $S(\rho)\perp S(\psi)$ and the decomposition is semi-orthogonal. To see the converse assume that for a given (stable) function $\psi \in S(\varphi)^{1/2}$ we have that $S(\varphi)^{1/2}=S(\rho)\oplus S(\psi)$. By the first part of the proof, the selection $Q'(w)=e^{iw}G(w+\pi)$ yields a stable generator $\psi' \in S(\varphi)^{1/2}$ such that Q' is the symbol of ψ' and $S(\phi)^{1/2}=S(\rho)\oplus S(\psi')$. Since $S(\psi)$ is also an orthogonal complement to $S(\rho)$ in $S(\varphi)^{1/2}$ we must have $S(\psi)=S(\psi')$ and $\hat{\psi}=K\widehat{\psi'}$ with K some 2π-periodic function. This implies that ψ has a symbol $Q(w)=e^{iw}G(w+\pi)K(2w)$ of type (5.13). Finally, to see that $0\neq K\in W$ it is sufficient to show that $K(2w)=e^{-iw}\Delta_{P,Q}(w+\pi)$. Let us define $H(w):=e^{-iw}P(w+\pi)K^{-1}(2w)$. It can be verified that the matrix

$$\begin{bmatrix} G(w) & G(w+\pi) \\ H(w) & H(w+\pi) \end{bmatrix},$$

is the inverse of $M_{P,Q}(w)$ defined in (5.4). In particular G can be represented by (5.5). Since $\Delta_{P,Q}\neq 0$ the functions $Q, Q(\cdot+\pi)$ do not vanish simultaneously at any $w\in\mathbb{T}$. Therefore we can conclude from (5.5) and the representation (5.13) that $K(2w)=e^{-iw}\Delta_{P,Q}(w+\pi)$.

♦

Using our last result we can always complement any generator by a semi-orthogonal counterpart. In particular, in the case of local spaces, the above gives us a method to construct a (minimal) compactly supported generator, as done in [Ch], by a proper selection of the periodic function K. Assume φ, ρ are stable and compactly supported where the symbol of ρ, denoted by P, is a trigonometric polynomial. Using (5.10), we see that the choice $K=[\hat{\rho},\hat{\rho}]$ in (5.13) leads to the following symbol

$$Q(w)=-e^{-iw}[\hat{\varphi},\hat{\varphi}](w+\pi)\overline{P(w+\pi)}. \tag{5.14}$$

It is easy to see that for compactly supported φ, ρ, the above symbol produces a complementary compactly supported wavelet.

We close this section with a few observations. Let φ be stable and two-scale refinable such that $S(\varphi)^{1/2} = S(\varphi) + S(\psi)$ is a decomposition where P, Q are the corresponding symbols of φ, ψ. In image coding applications perfect reconstruction subband filters banks derived from P, Q are used in discrete settings (see 7.3.2 in [Ma]). In many applications, one is not required to understand wavelet theory but simply to implement an efficient discrete filtering process. Furthermore, computational steps that seem necessary according to sampling theory, are ordinarily neglected (see the discussion in [Ma] pp. 257-258), but still good coding results are obtained. How can one explain this phenomenon? An interesting explanation can be given using the results of this section. As is well known in the signal processing community, the "perfect reconstruction decomposition condition" (5.3) is a property of the symbols P, Q and does not depend on the generator φ. Assume that condition (5.3) holds for two symbols P, Q and replace the generator φ by some other stable generator ρ_0 which need not be refinable. Then, by Theorem 5.3, the functions $\rho_1, \psi_1 \in S(\rho_0)^{1/2}$ that have P, Q as their symbols are a basis for $S(\rho_0)^{1/2}$. This means that (5.3) is a universal property of the symbols P, Q and the subband filters derived from them, regardless of the underlying functions. Furthermore, we will see in Section 6.5 that if in addition, the symbols P, Q have certain approximation properties, then the corresponding basis $\{\rho_1, \psi_1\}$ provides a decomposition which is meaningful in the context of wavelet theory, whenever $S(\rho_0)$ has good approximation properties.

In the following sections we construct (non-stationary) decompositions $S(\varphi)^{1/2} = S(\rho) + S(\psi)$ such that ψ has at least one zero moment. For example, in the superfunction construction this will be ensured by an orthogonality condition $S(\psi) \perp S(\phi)$ where ϕ provides approximation order ≥ 1 (Definition 6.1). Thus, the zero moment $\hat{\psi}(0) = 0$, together with sufficient decay ensure that the admissibility condition (see [Da], [Ma]) holds, i.e.

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty .$$

This means that our constructions also leads to a multitude of new examples for stationary wavelets, at least in the sense of continuous wavelet transforms. However, the non-stationary constructions are required whenever we want to replace the continuous transform by some type of discrete transform.

5.1 Non-stationary Superfunction wavelets

In this section we present constructions for non-stationary wavelets inspired by the superfunction techniques of [BDR1]. In our case the projection is done from a stationary reference space, but the superfunction and wavelet spaces are non-stationary. The abstract decomposition of Theorem 2.29 already tells us that, given a reasonable FSI space U, we can decompose it $U = U_1 \oplus W_1$ using a reference space V, with $len(V) < len(U)$, such that $W_1 \perp V$ and U_1, V are of the same length. The heuristics of the superfunction decompositions is justified in Section 6.4 where the approximation properties of the decomposition subspaces are discussed in detail.

Theorem 5.6 Let $U_0 \subset L_2(\mathbb{R}^d)$ be a (local) regular FSI space. Let V be a (local) FSI space with $len(V) = len(U_0)$. Then there exists a sequence of subspaces U_j, W_j, $j \geq 1$ such that

1. U_j and W_j are (local) regular FSI spaces with $len(U_j) = len(U_0)$, $len(W_j) = (2^d - 1) len(U_0)$,
2. $U_j \oplus W_j = U_{j-1}^{1/2}$,
3. $W_j \perp V$.

Proof Since dilation by 2^{-j}, $j \in \mathbb{N}$, preserves the property of (localness) regularity, $U_0^{1/2}$ is a (local) regular FSI of length $2^d len(U_0)$. By Theorem 2.29, $U_0^{1/2}$ can be decomposed into $U_0^{1/2} = U_1 \oplus W_1$ where $len(U_1) = len(V) = len(U_0)$, $W_1 \perp V$ and U_1, W_1 are (local) regular. By repeated decomposition we obtain an half-multiresolution with the required properties.

◆

Corollary 5.7 Let $U_0 \subset L_2(\mathbb{R}^d)$ be a (local) regular FSI space. Let V be a (local) FSI space with $len(V) = len(U_0)$. Then for any scale $J \in \mathbb{Z}$ we have the following formal wavelet decomposition

$$U_0^{2^{-J}} = \bigoplus_{j=-\infty}^{J-1} W_{J-j}^{2^{-j}}, \tag{5.15}$$

where $W_j = S(\Psi_j) \perp V$, are non-stationary (local) regular wavelet spaces with $len(W_j) = (2^d - 1) len(U_0)$.

As can be seen, the fact that we construct only half-multiresolutions is not a real restriction. By dilating the space U_0 and the construction to any given (fine) scale, it can be used to approximate any function in $L_2(\mathbb{R}^d)$ at any required level of accuracy. Also, for the special case $V = U_0$, with U_0 a refinable FSI space, the above construction recovers the classical stationary wavelets.

Since we ensured that each wavelet space W_j is regular, by [BDR2] Corollary 3.31, one can select for each $j \geq 1$ an orthonormal wavelet basis

$$\Psi_j = \left\{ \psi_{j,l} \middle| \ 1 \leq l \leq (2^d - 1) len(U_0) \right\},$$

for W_j. From the orthogonality $W_j \perp W_k$ for $j \neq k$, any selection of orthonormal bases Ψ_j for W_j provides, with the appropriate normalization, an orthonormal basis for $U_0^{2^{-J}}$, $J \in \mathbb{Z}$.

Next we discuss actual constructions that realize the decompositions of Theorem 5.6. There are two strategies we can employ: First we can follow the method of Theorem 2.29 by first constructing the superfunction spaces U_j using projection and then complementing them by the wavelet spaces W_j. Or we can construct W_j first using the orthogonality constraint $W_j \perp V$ and then complement by U_j. We start with the projection method and provide a construction for the generators of the spaces U_j in the local univariate case.

Theorem 5.8 Let $\phi, \varphi \in L_2(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq [0, m_\varphi]$, $\mathrm{supp}(\phi) \subseteq [0, m_\phi]$ such that $2 \leq m_\varphi, m_\phi \in \mathbb{N}$. If φ is stable, then there exists $\rho \in S(\varphi)^{1/2}$ such that $S(\rho) = P_{S(\varphi)^{1/2}} S(\phi)$ and

$$\left|\mathrm{supp}(\rho)\right| \leq \frac{5}{2} m_\varphi + m_\phi - \frac{5}{2}. \tag{5.16}$$

Proof Let us denote $\varphi_1 := \varphi(2\cdot), \varphi_2 := \varphi(2\cdot - 1)$. Since φ is stable $\{\varphi_1, \varphi_2\}$ is a basis for $S(\varphi)^{1/2}$. Using (2.11) and the method of proof of Theorem 2.28, we see that $P_{S(\varphi)^{1/2}} S(\phi) = S(\rho)$ where ρ is defined by

$$\hat{\rho} := \det G_1(\phi) \hat{\varphi}_1 + \det G_2(\phi) \hat{\varphi}_2, \tag{5.17}$$

with

$$G_1 := \begin{bmatrix} [\hat{\phi}, \hat{\varphi}_1] & [\hat{\phi}, \hat{\varphi}_2] \\ [\hat{\varphi}_2, \hat{\varphi}_1] & [\hat{\varphi}_2, \hat{\varphi}_2] \end{bmatrix}, \quad G_2 := \begin{bmatrix} [\hat{\varphi}_1, \hat{\varphi}_1] & [\hat{\varphi}_1, \hat{\varphi}_2] \\ [\hat{\phi}, \hat{\varphi}_1] & [\hat{\phi}, \hat{\varphi}_2] \end{bmatrix}.$$

Expanding (5.17) we have

$$\hat{\rho} = \left([\hat{\phi}, \hat{\varphi}_1][\hat{\varphi}_2, \hat{\varphi}_2] - [\hat{\phi}, \hat{\varphi}_2][\hat{\varphi}_2, \hat{\varphi}_1] \right) \hat{\varphi}_1 + \left([\hat{\varphi}_1, \hat{\varphi}_1][\hat{\phi}, \hat{\varphi}_2] - [\hat{\varphi}_1, \hat{\varphi}_2][\hat{\phi}, \hat{\varphi}_2] \right) \hat{\varphi}_2.$$

Since we assume $\operatorname{supp}(\varphi) \subseteq [0, m_\varphi]$, $\operatorname{supp}(\phi) \subseteq [0, m_\phi]$, we can bound the degrees of the following trigonometric polynomials

$$\left[\hat{\phi}, \hat{\varphi}_1\right](w) = \sum_{k=1-m_\phi}^{\lceil m_\varphi/2 \rceil - 1} \alpha_{1,k} e^{ikw}, \qquad \left[\hat{\varphi}_1, \hat{\varphi}_1\right](w) = \left[\hat{\varphi}_2, \hat{\varphi}_2\right](w) = \sum_{k=1-\lceil m_\varphi/2 \rceil}^{\lceil m_\varphi/2 \rceil - 1} \alpha_{2,k} e^{ikw},$$

$$\left[\hat{\phi}, \hat{\varphi}_2\right](w) = \sum_{k=1-m_\phi}^{\lfloor m_\varphi/2 \rfloor} \alpha_{3,k} e^{ikw}, \qquad \left[\hat{\varphi}_1, \hat{\varphi}_2\right](w) = \sum_{k=1-\lfloor m_\varphi/2 \rfloor}^{\lfloor m_\varphi/2 \rfloor} \alpha_{4,k} e^{ikw},$$

$$\left[\hat{\phi}, \hat{\varphi}_2\right](w) = \sum_{k=1-m_\phi}^{\lceil m_\varphi/2 \rceil} \alpha_{5,k} e^{ikw}.$$

After multiplication we obtain

$$\left[\hat{\phi}, \hat{\varphi}_1\right]\left[\hat{\varphi}_2, \hat{\varphi}_2\right] = \sum_{k=2-m_\phi-\lceil m_\varphi/2 \rceil}^{2\lceil m_\varphi/2 \rceil - 2} \beta_{1,k} e^{ikw}, \qquad \left[\hat{\phi}, \hat{\varphi}_2\right]\left[\hat{\varphi}_2, \hat{\varphi}_1\right] = \sum_{k=1-m_\phi+\lfloor m_\varphi/2 \rfloor}^{\lfloor m_\varphi/2 \rfloor + m_\varphi - 1} \beta_{2,k} e^{ikw},$$

$$\left[\hat{\varphi}_1, \hat{\varphi}_1\right]\left[\hat{\phi}, \hat{\varphi}_2\right] = \sum_{k=2-m_\phi+\lceil m_\varphi/2 \rceil}^{m_\varphi - 1} \beta_{3,k} e^{ikw}, \qquad \left[\hat{\varphi}_1, \hat{\varphi}_2\right]\left[\hat{\phi}, \hat{\varphi}_2\right] = \sum_{k=2-m_\phi+\lceil m_\varphi/2 \rceil}^{m_\varphi - 1} \beta_{4,k} e^{ikw},$$

Therefore

$$\hat{\rho}(w) = \hat{\varphi}_1 \sum_{k=2-m_\phi-\lceil m_\varphi/2 \rceil}^{\lfloor m_\varphi/2 \rfloor + m_\varphi - 1} \gamma_{1,k} e^{ikw} + \hat{\varphi}_2 \sum_{k=2-m_\phi-\lceil m_\varphi/2 \rceil}^{m_\varphi - 1} \gamma_{2,k} e^{ikw},$$

which finally leads to

$$\rho = \sum_{k=2-m_\phi-\lceil m_\varphi/2 \rceil}^{\lfloor m_\varphi/2 \rfloor + m_\varphi - 1} \gamma_{1,k} \varphi(2 \cdot + 2k) + \sum_{k=2-m_\phi-\lceil m_\varphi/2 \rceil}^{m_\varphi - 1} \gamma_{2,k} \varphi(2 \cdot + 2k - 1).$$

Thus,

$$\operatorname{supp}(\rho) \subseteq \left[1 - m_\varphi - \lfloor m_\varphi/2 \rfloor, m_\varphi/2 + \lceil m_\varphi/2 \rceil + m_\phi - \frac{3}{2}\right],$$

and $\left|\operatorname{supp}(\rho)\right| \le \frac{5}{2} m_\varphi + m_\phi - \frac{5}{2}$.

◆

Using methods mostly applied for non-uniform grids [LM], [LMQ] we can in some cases improve the estimate (5.16). That is, we can find a generator ρ', such that

$$\left|\operatorname{supp}(\rho')\right| < \frac{5}{2} m_\varphi + m_\phi - \frac{5}{2},$$

and $S(\rho') = S(\rho)$. The construction is as follows: First we construct an auxiliary wavelet $\psi \in S(\varphi)^{1/2}$ such that $S(\psi) \perp S(\phi)$ and then we construct ρ' as a complement for ψ such that $S(\varphi)^{1/2} = S(\rho') \oplus S(\psi)$.

We begin with the construction of the wavelet ψ. Assume that $\mathrm{supp}(\psi) \subseteq [0, y]$ where $y \in \mathbb{Z}$. Since we assumed that $\psi \in S(\varphi)^{1/2}$ we need to compute $2y - m_\varphi + 1$ unknowns $\{q_k\}_{k=0}^{2y-m_\varphi}$ where

$$\psi = \sum_{k=0}^{2y-m_\varphi} q_k \varphi(2\cdot - k).$$

Since $\mathrm{supp}(\phi) \subseteq [0, m_\phi]$ we have the following $y + m_\phi - 1$ constraints

$$\langle \psi, \phi(\cdot - j) \rangle = 0, \qquad j = 1 - m_\phi, \ldots, y - 1.$$

To have a non-trivial solution we must have the number of constraints $+1$ to be smaller or equal to the number of unknowns. Thus,

$$\underbrace{y + m_\phi - 1}_{\text{number of orthgonality constraints}} + 1 \le \underbrace{2y - m_\varphi + 1}_{\text{number of unknowns}}.$$

The smallest possible value $y = m_\varphi + m_\phi - 1$ leads to the following definition for ψ (up to a multiplicative constant)

$$\psi(x) = \det \begin{pmatrix} \langle \phi_{1-m_\phi}, \varphi_0 \rangle & \cdots & \langle \phi_{1-m_\phi}, \varphi_{2m_\phi + m_\varphi - 2} \rangle \\ \langle \phi_{2-m_\phi}, \varphi_0 \rangle & \cdots & \langle \phi_{2-m_\phi}, \varphi_{2m_\phi + m_\varphi - 2} \rangle \\ \vdots & & \vdots \\ \langle \phi_{m_\phi + m_\varphi - 2}, \varphi_0 \rangle & \cdots & \langle \phi_{m_\phi + m_\varphi - 2}, \varphi_{2m_\phi + m_\varphi - 2} \rangle \\ \varphi_0(x) & \cdots & \varphi_{2m_\phi + m_\varphi - 2}(x) \end{pmatrix},$$

where we have used $\phi_k := \phi(\cdot - k), \varphi_k := \varphi(2\cdot - k)$. We see that $q_k = (-1)^{m_\varphi - k} d_k$ where the minor d_k is defined by the Gram matrix

$$d_k := \det \begin{pmatrix} \phi_{1-m_\phi} & & \cdots & & \phi_{m_\phi + m_\varphi - 2} \\ \varphi_0 & \cdots & \varphi_{k-1} \quad \varphi_{k+1} & \cdots & \varphi_{2m_\phi + m_\varphi - 2} \end{pmatrix}. \tag{5.18}$$

From the above discussion we obtain the following result.

Theorem 5.9 Let $\phi, \varphi \in L_2(\mathbb{R})$ where φ is stable with $\mathrm{supp}(\varphi) \subseteq [0, m_\varphi]$, $\mathrm{supp}(\phi) \subseteq [0, m_\phi]$ such that $2 \le m_\varphi, m_\phi \in \mathbb{N}$. Assume the sequence $\{d_k\}_{k=0}^{m_\varphi + 2m_\phi - 2}$ defined by (5.18) is not identically zero. Then for $\psi := \sum_{k=0}^{m_\varphi + 2m_\phi - 2} q_k \varphi(2 \cdot - k)$, $q_k = (-1)^{m_\varphi - k} d_k$ we have that $S(\psi) \perp S(\phi)$ and $|\mathrm{supp}(\psi)| \le m_\varphi + m_\phi - 1$.

Example 5.10

1. Let $\varphi = \phi = N_m$ where N_m is the univariate B-spline of order m. It can be proved (see [LM], [LMK] for the general case of non-uniform knot sequences) that the B-splines fulfill the conditions of Theorem 5.9. Therefore, since $|\mathrm{supp}(N_m)| = m$, we recover the result of [Ch] that the support of the B-wavelet (minimally supported semi-orthogonal wavelet) is of size $2m-1$.
2. Let $\varphi = \phi = OM_4$ where $OM_4 := N_4 + N_4'' / 42$. Then it can be verified that $\psi_1 \in S(\varphi)^{1/2}$ defined by $\psi_1 = \sum_{k=0}^{10} q_k \varphi(2 \cdot - k)$ with $\{q_k\}$ given (up to a multiplicative constant) by the table below, is stable and fulfils the orthogonality condition $S(\psi) \perp S(\varphi)$.

k	q_k
0,10	-0.000347466
1,9	0.011939448
2,8	-0.099178639
3,7	0.374225526
4,6	-0.786638869
5	1.000000000

Even before the analysis of approximation properties is presented, it is easy to see that ψ_1 has all the required properties of a wavelet:

- The coefficients $\{q_k\}$ oscillate in sign,
- The coefficients $\{q_k\}$ as "high pass" filters have four vanishing moments,
- The function ψ_1 has four vanishing moments.

In fact, it can be verified that, with the right normalizations, the fifth (non vanishing) moment of $\{q_k\}$ or ψ_1 is closer to zero than the corresponding cubic B-spline wavelet with the same support size.

♦

Still, according to our theory, the wavelet ψ_1 constructed in Example 5.10 is only the first wavelet in a series of non-stationary wavelets that must be constructed if one wishes to decompose spaces of the type $S(OM_4)^{2^{-j}}$. The next wavelets in the sequence $\psi_2,\psi_3,\ldots$ still have four vanishing moments and as we will see, their fifth moment remains closer to zero than the cubic wavelet's fifth moment. In such examples, the price paid for removing the refinability property is that the support of the constructed wavelets might grow.

By constructing the wavelet first and assuming the conditions of Theorem 5.9, we can obtain a lower bound on the support of the complement superfunction. In such a case there exists a wavelet $\psi\in S(\varphi)^{1/2}$ with $|\mathrm{supp}(\psi)|\le m_\varphi+m_\phi-1$ such that $S(\psi)\perp S(\phi)$. Now we assume the conditions of Theorem 5.9 again, this time allowing ψ to play the role of the reference generator ϕ. This leads to the construction of a generator $\rho'\in S(\varphi)^{1/2}$ such that $S(\rho')\oplus S(\psi)=S(\varphi)^{1/2}$ with

$$|\mathrm{supp}(\rho')|\le m_\varphi+m_\psi-1\le m_\varphi+(m_\varphi+m_\phi-1)-1=2m_\varphi+m_\phi-2.$$

Observe that $S(\psi)\perp S(\phi)$ implies $P_{S(\varphi)^{1/2}}S(\phi)\subseteq S(\rho')$. But since $P_{S(\varphi)^{1/2}}S(\phi)$ is by Theorem 5.8 a local PSI space, we have using [BDR2] Corollary 2.6 that $S(\rho')=P_{S(\varphi)^{1/2}}S(\phi)$.

We end this section with a simple result on the relation between symbols of generators constructed using the superfunction projection method.

Theorem 5.11 Let $\psi\in S(\varphi)^{1/2}$ with $\varphi\in L_2(\mathbb{R})$ such that $Q\in\mathcal{W}$ is the symbol of ψ. Let ϕ generate a "reference" space $S(\phi)$ and assume

$$R(w)=\frac{1}{2}\sum_{k\in\mathbf{Z}}r_k e^{-ikw}\in\mathcal{W},\qquad \text{where } r_k=\langle\varphi(2\cdot-k),\phi\rangle.$$

Then, a necessary and sufficient condition for the orthogonality relation $S(\psi)\perp S(\phi)$ is

$$R(-w)Q(w)+R(-w-\pi)Q(w+\pi)=0,\qquad \forall w\in\mathbb{T}. \tag{5.19}$$

Proof The spaces $S(\phi)$ and $S(\psi)$ are orthogonal if and only if $[\hat{\phi},\hat{\psi}]\equiv 0$, or equivalently $\langle\psi,\phi(\cdot-l)\rangle=0$, $\forall l\in\mathbf{Z}$. This leads to the set of "convolution" conditions:

$$\begin{aligned}\langle\psi,\phi(\cdot-l)\rangle&=\Big\langle\sum_{k\in\mathbf{Z}}q_k\varphi(2\cdot-k),\phi(\cdot-l)\Big\rangle\\&=\sum_{k\in\mathbf{Z}}q_k\langle\varphi(2\cdot-k),\phi(\cdot-l)\rangle\end{aligned}$$

$$= \sum_{k \in \mathbb{Z}} q_k \left\langle \varphi\left(2 \cdot -(k-2l)\right), \phi \right\rangle$$
$$= \sum_{j \in \mathbb{Z}} q_{j+2l} \left\langle \varphi\left(2 \cdot - j\right), \phi \right\rangle = \sum_{j \in \mathbb{Z}} q_{j+2l} r_j = 0.$$

The left hand side of (5.19) is

$$R(-w)Q(w) + R(-w-\pi)Q(w+\pi) = \frac{1}{4}\left(\sum_j \sum_k r_j q_k e^{i(j-k)w} + \sum_j \sum_k r_j q_k e^{i(j-k)w} e^{-i\pi(j-k)} \right.$$
$$= \frac{1}{2} \sum_l \left(\sum_j r_j q_{j+2l} \right) e^{-i2lw}.$$

Hence (5.19) holds if and only if

$$\sum_j r_j q_{j+2l} = 0, \; l \in \mathbb{Z}.$$

5.2 Non-stationary Cascade wavelets

It is well known that the cascade operator can be used to obtain the $\mathcal{S}$ – refinable function of a subdivision scheme, or equivalently, a solution to a two-scale functional equation. Given a mask $P = \{p_k\}_{k\in Z^d}$, we define the **cascade operator** $\mathcal{C}$ by

$$\mathcal{C}f := \sum_{k\in\mathbb{Z}^d} p_k f(2\cdot -k).$$

Starting with an initial function $\rho_0 \in L_p(\mathbb{R}^d)$ one iterates $\rho_{j+1} = \mathcal{C}\rho_j$. A popular choice is $\rho_0 = N_2$, where N_2 is the tensor-product linear B-spline.

For our constructions we require the general results of [R2] on the cascade operator. Our application of these results is as follows. We have an initial generator ρ_0, which is possibly not refinable, but has good approximation properties. We would like to decompose the space $S(\rho_0)^{2^{-J}}$, corresponding to a certain scale J, to a sum of meaningful wavelet subspaces. The theory tells us that by carefully choosing an appropriate cascade operator and applying it to ρ_0, we obtain a sequence of generators $\rho_j = \mathcal{C}^j\rho_0$ such that:

1. The sequence $\{\rho_j\}$ converges in some (or all) p-metrics to the refinable function ϕ that is the "fixed point" of the operator $\mathcal{C}$.
2. The spaces $\{S(\rho_j)\}$ satisfy a nesting property $S(\rho_j) \subset S(\rho_{j-1})^{1/2}$.

Such a cascade sequence can be used to construct a "wavelet type" decomposition of the space $S(\rho_0)^{2^{-J}}$ in the following way. First we construct for each $j \geq 1$ a complement FSI space $S(\Psi_j)$ of length $2^d - 1$ such that $S(\rho_j) \oplus S(\Psi_j) = S(\rho_{j-1})^{1/2}$. Once such a non-stationary sequence of spaces is found we can (formally) decompose

$$S(\rho_0)^{2^{-J}} = \bigoplus_{j=1}^{\infty} S(\Psi_j)^{2^{-J+j}}.$$

The orthogonality $S(\Psi_j) \perp S(\Psi_k)$ for $j \neq k$ is not necessary, but simplifies the construction of stable bases (see Definition 2.18). Namely, we would like the set

$$\left\{2^{(J-j)d/2}\psi_{j,l}\left(2^{J-j}\cdot -k\right) \middle|\ j \geq 1, k \in \mathbb{Z}^d, \psi_{j,l} \in \Psi_j\right\}, \tag{5.20}$$

to be a stable basis for $S(\rho_0)^{2^{-J}}$. Indeed, we will construct wavelet generators Ψ_j that are a stable basis for $S(\Psi_j)$ with stability constants A_j, B_j where we ensure the uniform bound $A \le A_j \le B_j \le B$. Then, from the orthogonality condition, we can immediately derive that the set (5.20) is a stable basis for $S(\rho_0)^{2^{-J}}$, with uniform stability constants bounded by A, B.

As in the case of the superfunction construction of Section 5.1, we note that changing the approximation scale J does not require a different construction. We can dilate the construction and decompose any finer or coarser resolution.

In the following we use the notions of Sobolev spaces and approximation order that are defined in Chapter 6. The following is a simple form of Theorem 3.2.8 in [R2].

Theorem 5.12 [R2] Let $\phi \in W_p^m(\mathbb{R}^d)$ be refinable and a stable generator for $S(\phi)$. Denote by $\mathcal{C} := \mathcal{C}(\phi)$ the corresponding cascade operator. Let g be a bounded stable compactly supported function for which $\hat{\phi} - \hat{g} = O(|\cdot|^n)$ near the origin. If the shifts of g provide approximation order $\ge m$, then the cascade algorithm converges at the rate

$$\left\| \mathcal{C}^j g - \phi \right\|_{L_p(\mathbb{R}^d)} \le A_g 2^{-\min\{m,n\}j}.$$

We see that by a careful selection of the underlying refinable function ϕ we not only ensure convergence of the cascade process, but we can also estimate the convergence rate. For example, a typical application of Theorem 5.12 in our setting for the univariate case is as follows: Assume $\rho_0 = (I + D) N_m$ is a stable generator where D is some homogeneous differential operator of degree $n \le m-2$ such that

$$D = \sum_{k=1}^{n} \alpha_k \left(\frac{d}{dx} \right)^k. \tag{5.21}$$

Select the cascade operator $\mathcal{C}(N_m)$. Then, by (5.21) and the relation $\widehat{f^{(n)}} = (iw)^n \hat{f}$ we have

$$\left(\widehat{N_m} - \widehat{\rho_0} \right)(w) = \widehat{N_m}(w) \left(\sum_{k=1}^{n} \beta_k w^k \right).$$

We see that near the origin $\left| \left(\widehat{N_m} - \widehat{\rho_0} \right)(w) \right| \le A|w|$. As we shall see, each such ρ_0 provides the same approximation order as the B-spline and therefore the conditions of Theorem 5.12 are satisfied.

Example 5.13 Let $OM_4 := N_4 + N_4''/42$ ([BTU]) and denote $\mathcal{C} := \mathcal{C}(N_4)$. Then, since

$$\left(\widehat{N_4} - \widehat{OM_4}\right)(w) = w^2 \widehat{N_4}(w)/42 = O\left(|w|^2\right),$$

near the origin, we have

$$\left\|\mathcal{C}^j OM_4 - N_4\right\|_{L_p(\mathbb{R})} \le A 2^{-2j}.$$

In contrast to the convergence acceleration sought in [R2] using a smart choice of initial seed, there are cases where slow convergence is preferable. As we shall see in Section 6.5, this is the case whenever the initial function has better properties then the limit function. In such a case the first few levels of the cascade process have properties that are "close" to the properties of the initial function. This is useful in applications, since in practice only the first levels of the cascade are used.

Definition 5.14 Let ρ_0 be an initial function to the cascade process $\mathcal{C}$ defined by a refinable ϕ. Let $\rho_j = \mathcal{C}^j \rho_0$ and assume $\lim_{j\to\infty} \|\rho_j - \phi\|_{L_2(\mathbb{R}^d)} \to 0$. We call any sequence $\{\Psi_j\}$ such that $\{\rho_{j+1}, \Psi_{j+1}\}$ is a basis for $S(\rho_j)^{1/2}$ a **Cascade Wavelet** sequence.

For the rest of the section we assume that the masks of the cascade operators are finitely supported, hence also the corresponding refinable function. We now show that the cascade process interpolates the stability of the endpoints ρ_0, ϕ.

Theorem 5.15 Let $\rho_0 \in L_2(\mathbb{R}^d)$ be a stable compactly supported initial function and let $\mathcal{C}$ be a cascade process associated with a stable refinable $\phi \in L_2(\mathbb{R}^d)$. If $\lim_{j\to\infty} \|\rho_j - \phi\|_{L_2(\mathbb{R}^d)} \to 0$ where $\rho_j := \mathcal{C}^j \rho_0$, then there exist uniform stability constants $0 < \tilde{A} \le \tilde{B} < \infty$ such that for all $c \in l_2(\mathbb{Z}^d)$ and all $j \ge 0$

$$\tilde{A}\|c\|_{l_2(\mathbb{Z}^d)}^2 \le \left\|\sum_{k\in\mathbb{Z}^d} c_k \rho_j(\cdot - k)\right\|_{L_2(\mathbb{R}^d)}^2 \le \tilde{B}\|c\|_{l_2(\mathbb{Z}^d)}^2. \tag{5.22}$$

Proof From Theorem 2.20 we know that for each ρ_j, $j \ge 0$, the sharp stability constants A_j, B_j in (2.9) are given by the min/max values of the auto-correlation polynomials $[\hat{\rho}_j, \hat{\rho}_j]$. By Lemma 2.13 we have the convergence $A_j \to A$, $B_j \to B$ where A, B are the sharp stability constants of ϕ. Thus, since a converging sequence is bounded, we need only prove that each $A_j > 0$.

Let $P(w) = \sum_{k \in \mathbf{Z}^d} p_k e^{-iwk}$ be the trigonometric polynomial corresponding to the finite mask of the cascade operator $\mathcal{C}$. We argue that since ϕ is stable, we must have

$$\sum_{e \in E_d} \left| P(w + \pi e) \right|^2 > 0, \quad \forall w \in \mathbb{T}^d, \tag{5.23}$$

where we have used the lattice decomposition (5.2). Indeed, assume that $P(w_0 + \pi e) = 0$, $\forall e \in E_d$, for some $w_0 \in \mathbb{T}^d$. Then, by the refinability of ϕ

$$\begin{aligned} \left[\hat{\phi}, \hat{\phi}\right](2w_0) &= \sum_{k \in \mathbf{Z}^d} \left| \hat{\phi}(2w_0 + 2\pi k) \right|^2 \\ &= \sum_{k \in \mathbf{Z}^d} \left| P(w_0 + \pi k) \right|^2 \left| \hat{\phi}(w_0 + \pi k) \right|^2 \\ &= \sum_{e \in E_d} \left| P(w_0 + \pi e) \right|^2 \left[\hat{\phi}, \hat{\phi}\right](w_0 + \pi e) = 0. \end{aligned}$$

Since ϕ is compactly supported, $\left[\hat{\phi}, \hat{\phi}\right]$ is a trigonometric polynomial and by Theorem 2.20, this contradicts the stability of ϕ. Equipped with (5.23) we can now apply Lemma 5.2 inductively to obtain that each $A_j > 0$. Since $A_j \to A > 0$, there must exist $\tilde{A} > 0$ such that $A_j \geq \tilde{A} > 0$ for $j \geq 1$, hence the uniform stability (5.22).

♦

Corollary 5.16 Under the assumptions of Theorem 5.15, assume further that $d = 1$, $\bigcap_{j=-\infty}^{\infty} S(\phi)^{2^{-j}} = \{0\}$ and let $\{\psi_j\}$ be a univariate Cascade wavelet sequence where $S(\rho_{j+1}) \oplus S(\psi_{j+1}) = S(\rho_j)^{1/2}$ for all $j \geq 0$. Assume that for each $j \geq 1$ the symbols $\{Q_j\}$ of the wavelets $\{\psi_j\}$ meet the following conditions

1. $Q_j \in \mathcal{W}$ with $\|Q_j\|_{C(\mathrm{T})} \leq B' < \infty$.
2. $|Q_j(w)|^2 + |Q_j(w + \pi)|^2 \geq A' > 0$, $\forall w \in \mathbb{T}$,

Then for any $J \in \mathbb{Z}$ the dilated non-stationary wavelet set $\left\{2^{(J-j)/2} \psi_j\left(2^{J-j} \cdot - k\right)\right\}_{j \geq 1, k \in \mathbf{Z}}$ is a stable basis for $S(\rho_0)^{2^{-J}}$.

Proof This is an immediate consequence of Lemma 5.2 and Theorem 5.15.

♦

Thus, using the cascade operator together with a careful choice of non-stationary Cascade wavelet sequence, we can provide a stable decomposition of a (possibly non-refinable) approximation space $S(\rho_0)^{2^{-J}}$. Next we use the general tools presented at the beginning of the chapter to construct such a non-stationary Cascade wavelet sequence for a given univariate cascade sequence.

Let $\rho_0 \in L_2(\mathbb{R})$ be a stable compactly supported initial function and P the finitely supported cascade mask. Using the construction of (5.11) we define for $j \geq 1$

$$G_j = \frac{\left[\hat{\rho}_{j-1}, \hat{\rho}_{j-1}\right]}{\left[\hat{\rho}_j, \hat{\rho}_j\right](2\cdot)} \overline{P}. \tag{5.24}$$

Since $\left[\hat{\rho}_j, \hat{\rho}_j\right] > 0$ is a trigonometric polynomial for $j \geq 0$, by Wiener's lemma [K], we have that $G_j \in \mathbb{W}$ for each $j \geq 1$. As shown in Theorem 5.5, any wavelet ψ_j such that $S(\rho_{j+1}) \oplus S(\psi_{j+1}) = S(\rho_j)^{1/2}$ has a two-scale relation of the form

$$Q_j(w) = e^{iw} G_j(w+\pi) K_j(2w), \qquad 0 \neq K_j \in \mathbb{W}. \tag{5.25}$$

Recall that in this local setting we can use the construction (5.14) to choose $\{K_j\}$ such that $\{Q_j\}$ are trigonometric polynomials and thus construct $\{\psi_j\}$ with compact support. We therefore obtain,

Theorem 5.17 Let $\rho_0 \in L_2(\mathbb{R})$ be a stable compactly supported initial function. Let $\mathcal{C}$ be a cascade process defined by a compactly supported mask P. Assume that $\phi \in L_2(\mathbb{R})$, the refinable function corresponding to $\mathcal{C}$, is stable and that $\bigcap_{j=-\infty}^{\infty} S(\phi)^{2^{-j}} = \{0\}$. If $\rho_j := \mathcal{C}^j \rho_0$ converge to ϕ then there exist wavelets $\{\psi_j\}_{j=1}^{\infty}$ such that:

1. $S(\rho_{j+1}) \oplus S(\psi_{j+1}) = S(\rho_j)^{1/2}$ for all $j \geq 0$.

2. $\{\psi_j\}_{j=1}^{\infty}$ are compactly supported with a uniform bound on their support.

3. For any $J \in \mathbb{Z}$ the dilated non-stationary wavelet set $\left\{2^{(J-j)/2}\psi_j\left(2^{J-j}\cdot -k\right)\right\}_{j\geq 1, k\in\mathbf{Z}}$ are a stable basis for $S(\rho_0)^{2^{-J}}$.

Proof This result is a direct consequence of our analysis so far. For each $j \geq 1$ we select $\hat{\psi}_j = Q_j(\cdot/2)\hat{\rho}_{j-1}(\cdot/2)$ where

$$Q_j(w) := e^{iw}\left[\hat{\rho}_{j-1}, \hat{\rho}_{j-1}\right](w)\bar{P}(w).$$

This is equivalent to the selection $K_j = \left[\hat{\rho}_j, \hat{\rho}_j\right]^{-1}$ in (5.25). We already know that ψ_j is a semi-orthogonal complement to ρ_j such that $S(\rho_j) \oplus S(\psi_j) = S(\rho_{j-1})^{1/2}$. Also, observe that since the auto-correlation $\left[\hat{\rho}_{j-1}, \hat{\rho}_{j-1}\right]$ and P are trigonometric polynomials, so is Q_j. Thus, $\{\psi_j\}$ all have compact support. Furthermore, we can uniformly bound their support because of the convergence $\rho_j \to \phi$ using a finitely supported cascade mask. It remains to show that there exist uniform stability bounds $0 < A \leq B < \infty$, such that for any $c \in l_2(\mathbb{Z})$

$$A\|c\|_{l_2(\mathbb{Z})}^2 \leq \left\|\sum_{k \in \mathbb{Z}} c_k \psi_j(\cdot - k)\right\|_{L_2(\mathbb{R})}^2 \leq B\|c\|_{l_2(\mathbb{Z})}^2.$$

Applying Theorem 5.15 we have that there exist uniform bounds $0 < \tilde{A} \leq \tilde{B} < \infty$ such that for each $j \geq 0$ we have

$$\tilde{A} \leq \left[\hat{\rho}_j, \hat{\rho}_j\right] \leq \tilde{B}. \tag{5.26}$$

We now use (5.26) to uniformly bound the polynomials $\{Q_j\}$

$$\|Q_j(w)\|_\infty \leq \left\|\left[\hat{\rho}_{j-1}, \hat{\rho}_{j-1}\right]\right\|_\infty \|P\|_\infty \leq \tilde{B}\|P\|_\infty =: B < \infty. \tag{5.27}$$

Next we use (5.26) together with (5.23) in the following

$$\begin{aligned} |Q_j(w)|^2 + |Q_j(w+\pi)|^2 &= \left(\left[\hat{\rho}_{j-1}, \hat{\rho}_{j-1}\right](w)\right)^2 |P(w)|^2 + \left(\left[\hat{\rho}_{j-1}, \hat{\rho}_{j-1}\right](w+\pi)\right)^2 |P(w+\pi)|^2 \\ &\geq \tilde{A}^2\left(|P(w)|^2 + |P(w+\pi)|^2\right) \geq A > 0. \end{aligned} \tag{5.28}$$

Equipped with the estimates (5.27) and (5.28) we now apply Corollary 5.16 to derive the uniform stability of the non-stationary wavelet sequence $\{\psi_j\}$ with the uniform bounds A, B.

♦

6 Approximation properties

In classical refinable setting, it is a standard practice to derive (linear) approximation properties of wavelets from the approximation properties of the scaling functions. In contrast, we have constructed in the previous chapter half-multiresolutions composed of a nested sequence of non-stationary spaces, beginning with some non-refinable shift invariant space. In this section we focus most of our attention on inheritence of approximation properties from the initial non-refinable shift invariant space to the subsequent nested spaces of the half-multiresolution. Our construction only becomes meaningful if the nested sequence of spaces share uniform approximation properties. Specifically, we provide simultaneous estimates using uniform constants for approximations of functions from these spaces. Exactly as in classical refinable setting, this analysis provides a mean to determine the (linear) approximation properties of the non-stationary wavelet spaces that are the difference spaces of the half-multiresolutions.

First we introduce the classical smoothness spaces in which our estimates take place. Then, for both the general case $1 \le p \le \infty$ and the Hilbert space case $p = 2$ we discuss known results from the well researched topic of approximation from shift invariant spaces and also present some new results for simultaneous estimates and error estimates using optimal constants. We then proceed to the main part of this section and make the Superfunction and Cascade wavelet constructions of Sections 5.1 and 5.2 meaningful by showing that the constructed half-multiresolutions inherit the approximation properties of the initial non-refinable shift invariant space.

Throughout this chapter we use the standard notation for the error of approximation

$$E(f,V)_X := \inf_{g \in V} \|f - g\|_X ,$$

where $V \subseteq X$ is a closed subspace of a Banach space X.

6.1 Smoothness spaces

1. Sobolev spaces $W_p^m(\mathbb{R}^d)$

These basic smoothness spaces relate to differentiation. In this work we denote by $W_p^m(\mathbb{R}^d)$ the collection of functions for which

a. The partial derivatives $D^\alpha f$ are in $C^{m-|\alpha|-1}(\mathbb{R}^d)$ for $1 \le |\alpha| < m-1$ and are absolutely continuous in each variable for $|\alpha| = m-1$.

b. $D^\alpha f \in L_p(\mathbb{R}^d)$ for $|\alpha| \le m$.

The Sobolev spaces are traditionaly used to evaluate the quality of linear approximation. Also, since more general functions can be approximated by smooth functions, estimates using Sobolev norms are tools for error estimates using other smoothness measures. The Sobolev semi-norm and norm are

$$|f|_{W_p^m} := \sum_{|\alpha|=m} \|D^\alpha f\|_{L_p(\mathbb{R}^d)}, \qquad \|f\|_{W_p^m(\mathbb{R}^d)} := \|f\|_{L_p(\mathbb{R}^d)} + |f|_{W_p^m(\mathbb{R}^d)}.$$

2. Potential spaces $H^r(\mathbb{R}^d)$

For $p=2$, there is a known generalization of Sobolev spaces which applies to "intermediate" smoothness parameters. The spaces $H^r(\mathbb{R}^d)$ are defined for $r \in \mathbb{R}_+$ by

$$H^r := \left\{ f \in L_2(\mathbb{R}^d) \;\middle|\; \|f\|_{H^r} := (2\pi)^{-d} \left\| (1+|\cdot|)^r \hat{f} \right\|_{L_2(\mathbb{R}^d)} < \infty \right\}.$$

For any $f \in H^r(\mathbb{R}^d)$ the semi-norm is defined by

$$|f|_{H^r(\mathbb{R}^d)} := (2\pi)^{-d} \left\| |\cdot|^r \hat{f} \right\|_{L_2(\mathbb{R}^d)}.$$

3. Moduli of smoothness

Here we only define and use the Moduli of smoothness over $\mathbb{R}^d$ (see [DL] for the definition over general domains and general properties). From the first difference $\Delta_h(f)(x) := f(x+h) - f(x)$, $h \in \mathbb{R}^d$ we define higher order differences by $\Delta_h^r := \Delta_h(\Delta_h^{r-1})$. It follows from the binomial theorem that

$$\Delta_h^r(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh).$$

For each $0 < t \in \mathbb{R}$ and $1 \le p \le \infty$, the r-th modulus of smoothness of $f \in L_p(\mathbb{R}^d)$ is defined by

$$\omega_r(f,t)_p := \sup_{0 \le |h| \le t} \|\Delta_h^r(f,\cdot)\|_{L_p(\mathbb{R}^d)}.$$

6.2 L_p approximation from shift invariant spaces

Definition 6.1 A closed subspace $V \subset L_p(\mathbb{R}^d)$ is said to provide L_p **approximation order** m if for any function $f \in W_p^m(\mathbb{R}^d)$

$$E(f,V^h)_p \le C(V,f)h^m. \tag{6.1}$$

Most results on (linear) approximation from shift invariant spaces use the Sobolev semi-norm of the approximated function for the constant (6.1), leading to a Jackson-type estimate

$$E(f,V^h)_p \le C(V)h^m |f|_{W_p^m}. \tag{6.2}$$

Estimates using the Sobolev semi-norm are important for applications. Also, they correspond to polynomial reproduction. Although polynomials are not in $L_p(\mathbb{R}^d)$, observe that if f is a polynomial in Π_{m-1} then $|f|_{W_p^m} = 0$ and our approximation becomes a reproduction. If V is an SI space such that $V = S(\Phi)$, we sometimes change notation and replace the constant $C(V)$ by C_Φ.

Definition 6.2 An SI space $S(\Phi) \subset L_p(\mathbb{R}^d)$ is said to provide **controlled** L_p **approximation of order** m, if it provides approximation order m such that for any $h > 0$ and $f \in W_p^m(\mathbb{R}^d)$ there exist coefficients $c_{\Phi,h} := \{c_{h,\phi,k}\}_{\phi\in\Phi,k\in\mathbb{Z}^d}$ so that

1. $\left\| f - h^{-d/p} \sum_{\phi\in\Phi} \sum_{k\in\mathbb{Z}^d} c_{h,\phi,k}\phi(h^{-1}x-k) \right\|_{L_p(\mathbb{R}^d)} \le C_1 h^m |f|_{W_p^m(\mathbb{R}^d)}$,

2. $\|c_{\Phi,h}\|_{l_p(\Phi\times\mathbb{Z}^d)} \le C_2 \|f\|_{L_p(\mathbb{R}^d)}$ where the constant C_2 does not depend on h.

Definition 6.3 A finite generating set Φ of a FSI space is said to satisfy the **Strang-Fix (SF)** conditions of order m if there exist finitely supported sequences $\beta_\phi \in l_\infty(\mathbb{Z}^d)$, $\phi \in \Phi$, such that for the function $\varphi \in S(\Phi)$,

$$\varphi = \sum_{\phi\in\Phi} \sum_{k\in\mathbb{Z}^d} \beta_{\phi,k}\phi(\cdot - k),$$

the following conditions hold

$$\hat{\varphi}(0) \neq 0 \text{ and } D^{\alpha}\hat{\varphi}(2\pi k) = 0 \text{ for all } k \in \mathbb{Z}^d \setminus 0 \text{ and } |\alpha| < m. \tag{6.3}$$

The function φ is sometimes called a **superfunction.** We shall see that if the SF conditions hold, then the PSI space $S(\varphi)$ provides, up to a constant, the same approximation order of the larger FSI space $S(\Phi)$.

The following is a very general result of Jia and Lei [JL] that characterizes controlled approximation from shift invariant spaces. It treats the cases of $1 \leq p \leq \infty$ and global support under a very weak "variation" assumption. We first define the notion of normal functions and then proceed to quote their result.

Definition 6.4 [JL] A multivariate measurable function is called **normal** if it is locally integrable and for any $x \in \mathbb{R}^d$,

$$\phi(x) = \lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} \phi(y)\,dy, \qquad B_\varepsilon(x) := \left\{ y \in \mathbb{R}^d \,\middle|\, \|y - x\| < \varepsilon \right\}.$$

Theorem 6.5 [JL] Let Φ be a finite collection of normal functions in $\mathbb{E}_m(\mathbb{R}^d)$ (see Definition 2.8). Then Φ provides controlled L_p approximation of order m if and only if Φ satisfies the SF conditions.

Example 6.6 Here are two examples for univariate, compactly supported generators which provide L_p approximation order and are important in applications:

1. N_m, B-splines - These generators can be defined for each order $m \geq 1$ by their Fourier transform

$$\widehat{N_m}(w) = \left(\frac{1 - e^{-iw}}{iw} \right)^m. \tag{6.4}$$

For each order m they are piecewise polynomial with degree $m-1$, have C^{m-2} smoothness and support size m. From (6.4) it is easy to see that for the B-spline of order m the SF conditions (6.3) of order m hold and so they provide (controlled) approximation order m.

2. OM_m, O-Moms (Optimal Maximum Order and Minimal Support) - These generators ([BTU]) are designed using B-Splines, but have a certain optimal quality. For each order $m \geq 1$, the O-Moms function of order m, OM_m can be defined as a result of a differential operator $I + D_m$ on the B-spline N_m, where D_m is homogeneous with $|D_m| \leq m-1$.

It is easy to see that for any differential operator of the type $I+D$, the resulting $(I+D)N_m$ is piecewise polynomial with degree $m-1$ and support size m. Also, since the SF conditions still hold, OM_m provides approximation order m. The 0-Moms sequence of optimal generators is defined in the following way: for the first two orders $m=1,2$, the operators $D_1, D_2 = 0$ and so $OM_1 = N_1$, $OM_2 = N_2$. Assume the differential operators have the following form

$$I + D_m = P_m\left(\frac{d}{dx}\right) := \sum_{k=0}^{m-1} p_{m,k} \frac{d^k}{dx^k}, \qquad p_{m,0} = 1.$$

Then sequence $\{OM_m\}$ is defined recursively by

$$P_{m+2} = P_{m+1} + \frac{C_{m+1}^2}{C_m^2} x^2 P_m, \qquad C_m := \frac{1}{|P_m(0)|}\sqrt{\sum_{k\neq 0}\left|\frac{P_m(2\pi ik)}{(2\pi ik)^m}\right|^2}.$$

It is easy to see that P_{2n-1}, P_{2n} are of degree $2n-2$ and therefore the O-Moms functions are only continuous for the even orders. For example,

$$OM_4 = N_4 + \frac{1}{42}N_4^{(2)},$$
$$OM_6 = N_6 + \frac{1}{33}N_6^{(2)} + \frac{1}{7920}N_6^{(4)}.$$

Next we present some basic univariate Strang-Fix theory and prove a simpler version of Theorem 6.5, by repeating the approach of [DL]. However, we make some new observations that will become useful in our non-stationary Strang-Fix type result Theorem 6.10. We begin with the following result which connects polynomial reproduction to the SF conditions, using the Poisson Summation formula (2.16) as the main tool.

Lemma 6.7 Let $\phi \in \mathbb{E}_m(\mathbb{R})$ such that ϕ satisfies the SF conditions of order m and the summation conditions of order m (see Definition 2.31). Then there exist polynomials $P_l(\phi) := P_l$ of degrees $0 \le l < m$, such that:

1. $\sum_{k\in\mathbb{Z}} P_l(k)\phi(x-k) = x^l$, (6.5)
2. $P_l(0) = F_l\left(\hat{\phi}(0),\ldots,\hat{\phi}^{(l)}(0)\right)$, (6.6)

where $F_l : \mathbb{R}^{l+1} \to \mathbb{R}$ is a multivariate polynomial independent of ϕ.

Proof We assume without loss of generality the normalization $\hat{\phi}(0)=1$. We require the following (formal) formula relating shift invariance moments and derivatives of Fourier transforms.

$$\widehat{(\cdot)^l f(x-\cdot)}(y)=i^l \frac{d^l}{dy^l}\left(e^{-ixy}\hat{f}(-y)\right). \tag{6.7}$$

Since for ϕ the SF conditions hold, using (6.7) we have for any $1 \le l < m$

$$\widehat{(\cdot)^l \phi(x-\cdot)}(2\pi k)=\begin{cases}\hat{\phi}(0)x^l+R_{l-1}(x) & k=0,\\ 0 & \text{else,}\end{cases}$$

where $R_{l-1}\in\Pi_{l-1}$. We now define for a fixed $x\in\mathbb{R}$ the function $f(t):=t^l\phi(x-t)$. We obtain by equating the Poisson summation formula (2.16) for f at the origin

$$\sum_{k\in\mathbb{Z}} k^l\phi(x-k)=\sum_{k\in\mathbb{Z}}\widehat{(\cdot)^l\phi(x-\cdot)}(2\pi k)=\hat{\phi}(0)x^l+R_{l-1}(x)=x^l+R_{l-1}(x).$$

We also observe that by (6.7) for each $1\le l<m$ we have $R_{l-1}(x)=\sum_{j=0}^{l-1}c_{l-1,j}x^j$, where each of the coefficients $c_{l-1,j}$ is a linear multivariate function in the parameters $\left\{\hat{\phi}^{(r)}(0)\right\}_{r=1}^{l}$. We now proceed by induction. Since ϕ satisfies the SF conditions of order 0, we have for $l=0$

$$\sum_{k\in\mathbb{Z}}\phi(x-k)=\sum_{k\in\mathbb{Z}}\hat{\phi}(2\pi k)e^{i2\pi kx}=\hat{\phi}(0)=1.$$

Thus, we can choose $P_0\equiv 1$, independently of ϕ, satisfying (6.6) for $l=0$. Assume that (6.6) holds for $0\le l\le s-1$. Using $R_{s-1}(x)=\sum_{l=0}^{s-1}c_{s-1,l}x^l$, we define $P:=\sum_{l=0}^{s-1}c_{s-1,l}P_l$ and $P_s(x):=x^s-P(x)$.

1. First we observe that for P_s we have that $P_s(0)=F_s\left(\hat{\phi}'(0),\ldots,\hat{\phi}^{(s)}(0)\right)$, where F_s is a multivariate polynomial which does not depend on ϕ. This is true because each previous P_l and each coefficient $c_{j-1,l}$, $0\le l\le s-1$ are of this type.
2. For the polynomial P we have from the inductive assumption for a fixed x

$$\sum_{k\in\mathbb{Z}} P(k)\phi(x-k)=\sum_{k\in\mathbb{Z}}\left(\sum_{l=0}^{s-1}c_{s-1,l}P_l(k)\right)\phi(x-k)$$
$$=\sum_{l=0}^{s-1}c_{s-1,l}\sum_{k\in\mathbb{Z}}P_l(k)\phi(x-k)$$
$$=R_{s-1}(x).$$

3. Finally we obtain for a fixed x

$$\sum_{k\in\mathbb{Z}} P_s(k)\phi(x-k)=\sum_{k\in\mathbb{Z}} k^s\phi(x-k)-\sum_{k\in\mathbb{Z}} P(k)\phi(x-k)$$
$$=x^s+R_{s-1}(x)-R_{s-1}(x)$$
$$=x^s.$$

♦

The above polynomial reproduction property of generators that satisfy the SF conditions leads to approximation results. Given $m\ge 0$, we now assume the following on a univariate function ϕ:

1. $supp(\phi)\subseteq[-L,L]$,
2. $\phi\in L_\infty(\mathbb{R})$,
3. ϕ satisfies the SF conditions of order m (see (6.3)),
4. ϕ satisfies the Poisson summation conditions of order m (see Definition 2.31).

Next we show that each $P\in\Pi_{m-1}$ has a representation

$$P=\sum_{k=-\infty}^{\infty}a_k(P)\phi(\cdot-k). \tag{6.8}$$

where the functionals a_k are

$$a_k(P)=\sum_{l=0}^{m-1}\frac{P^{(l)}(0)}{l!}P_l(k)=a_0(P(\cdot+k)). \tag{6.9}$$

Indeed, let $P(x)=\sum_{l=0}^{m-1}\alpha_l x^l$. Then using (6.5) and the Taylor expansion of P at the origin we can define

$$a_0(P):=\sum_{l=0}^{m-1}\alpha_l P_l(0)=\sum_{l=0}^{m-1}\frac{P^{(l)}(0)}{l!}P_l(0).$$

In general, it is easy to see that by defining for any $k\in\mathbb{Z}$

$$a_k(P):=\sum_{l=0}^{m-1}\alpha_l P_l(k)=\sum_{l=0}^{m-1}\frac{P^{(l)}(0)}{l!}P_l(k),$$

we obtain the representation (6.8).

Let Ω be any domain containing 0 in its interior. Using (6.5) and Theorem 2.7 of chapter 4 in [DL], we can estimate the norm of the functional $a_0 : \Pi_{m-1}(\Omega) \to \mathbb{R}$ defined in (6.9)

$$|a_0(P)| \le \sum_{l=0}^{m-1} \left| \frac{P^{(l)}(0)}{l!} \right| |P_l(0)| \le C\left(\Omega, m, |\hat{\phi}(0)|, \ldots, |\hat{\phi}^{(m-1)}(0)|\right) \|P\|_1(\Omega). \tag{6.10}$$

By Hahn-Banach we can extend the functional a_0 from the subspace $\Pi_{m-1}(\Omega)$ to a functional over $L_1(\Omega)$ with the same norm. Thus, there exists a function G, bounded on Ω, such that

$$a_0(f) = \int_\Omega f(t) G(t)\, dt, \qquad f \in L_1(\Omega), \quad \|G\|_{L_\infty(\Omega)} \le C.$$

We extend G to be zero outside of Ω and obtain

$$a_k(f) = a_0(f(\cdot + k)) = \int_\Omega f(t+k) G(t)\, dt = \int_{\mathbb{R}} f(t) G(t-k)\, dt.$$

In what follows we take $\Omega = \operatorname{supp}(\phi) = [-L, L]$ and denote $\Omega_k = [-L, L] + k$. We now define the **quasi-interpolation operator**

$$Q_h(f, x) = \sum_{k \in \mathbb{Z}} a_k(f(h\cdot)) \phi\left(\frac{x}{h} - k\right). \tag{6.11}$$

Clearly $Q_h(P) = P$ for each $P \in \Pi_{m-1}$. We can also define the action of the operator Q_h using a convolution kernel,

$$Q_h(\phi)(f) := Q_h(f, x) = \frac{1}{h} \int_{\mathbb{R}} f(t) K_h(t, x)\, dt, \tag{6.12}$$

where the kernel is

$$K_h(\phi) := K_h(t, x) = \frac{1}{h} \sum_{k \in \mathbb{Z}} G\left(\frac{t}{h} - k\right) \phi\left(\frac{x}{h} - k\right). \tag{6.13}$$

Since both G, ϕ are supported on $[-L, L]$, the k-th term in (6.13) is non-zero only if x/h and t/h are in Ω_k. This can occur for at most $2L$ values of k. Hence with $M := \|\phi\|_\infty$ we have

$$|K_h(t, x)| \begin{cases} = 0 & |x - t| \ge 2Lh, \\ \le 2LCMh^{-1} & x, t \in \mathbb{R}. \end{cases} \tag{6.14}$$

It follows that $\int_{\mathbb{R}}|K_h(t,x)|dt \le 4L^2CM := C_1$. With similar estimates for $\int_{\mathbb{R}}|K_h(t,x)|dx$. We obtain

Theorem 6.8 [DL] The quasi-interpolation operator $Q_h(\phi)$ defined by (6.11) with the kernel $K_h(\phi)$ of (6.13) reproduces polynomials of degree $< m$. For $1 \le p \le \infty$ $Q_h(\phi)$ is a bounded operator on L_p with norm $\le C_1$.

Proof It is obvious that for each $h > 0$, Q_h is a linear operator. Also, it is easy to see that $\|Q_h\|_\infty, \|Q_h\|_1 \le C_1$. By Riesz-Thorin Q_h is a bounded operator on L_p with norm $\le C_1$.

♦

We shall require the following lemma for our estimates of approximation from PSI spaces. We use the same approach of [DL] to obtain a slightly more general result. The claim is very simple: it says that integrating remainders over a finite volume can enlarge the remainder by a fixed multiplicative constant which only depends on the size of the volume.

Lemma 6.9 Let $f \in W_p^n(\mathbb{R})$ and denote by $R_n(x,t)$ the remainder of the Taylor expansion of f of degree $n-1$ about the point $x \in \mathbb{R}$.Let $K(t,x)$ be a kernel such that

1. For $1 \le p < \infty$

$$|K(t,x)| \begin{cases} = 0 & |x-t| \ge C_1, \\ \le C_2 & (t,x) \in \mathbb{R}^2. \end{cases} \tag{6.15}$$

2. For $p = \infty$

$$|K(t,x)| \le C(|t-x|+1)^{-\alpha}, \qquad \alpha > n+1. \tag{6.16}$$

Define the dilated kernels $K_h := h^{-1}K(h^{-1}\cdot)$, $h > 0$. Then we have the estimate

$$\left\| \int_{\mathbb{R}} R_n(x,t) K_h(t,x) dt \right\|_p \le Ch^n \left\| f^{(n)} \right\|_p .$$

Proof

1. We first treat the case $1 \le p < \infty$ using the compactly supported band of the kernel. From (6.15) it is easy to see that

$$|K_h(t,x)| \begin{cases} = 0 & |x-t| \ge C_1 h, \\ \le C_2 h^{-1} & x,t \in \mathbb{R}. \end{cases} \tag{6.17}$$

For $|t-x|\le C_1 h$ we have

$$\begin{aligned}|R_n(x,t)| &= \left|\int_x^t \frac{(t-u)^{n-1}}{(n-1)!} f^{(n)}(u)\,du\right| \\ &\le \frac{(C_1 h)^{n-1}}{(n-1)!} \int_{x-C_1 h}^{x+C_1 h} \left|f^{(n)}(u)\right| du \\ &\le C h^{n-1/p} \left(\int_{x-C_1 h}^{x+C_1 h} \left|f^{(n)}(u)\right|^p du \right)^{1/p},\end{aligned}$$

where we have used the Hölder inequality for the last step. Using the dilated bound (6.17) we can proceed with

$$\begin{aligned}\left\|\int_{\mathbb{R}} R_n(x,t) K_h(t,x)\,dt\right\|_p^p &= \int \left| \int_{|x-t|\le C_1 h} R_n(x,t) K_h(t,x)\,dt \right|^p dx \\ &\le \int \left| C_2 h^{-1} \int_{|x-t|\le C_1 h} |R_n(x,t)|\,dt \right|^p dx \\ &\le C_3 \int \left| h^{-1} \int_{|x-t|\le C_1 h} h^{n-1/p} \left(\int_{x-C_1 h}^{x+C_1 h} \left|f^{(n)}(u)\right|^p du \right)^{1/p} dt \right|^p dx \\ &\le C_4 \int \left| h^{n-1/p} \left(\int_{x-C_1 h}^{x+C_1 h} \left|f^{(n)}(u)\right|^p du \right)^{1/p} \right|^p dx \\ &\le C_5 h^{np} \int \left|f^{(n)}(x)\right|^p dx.\end{aligned}$$

2. Assume $p=\infty$ and that (6.16) holds. Then for any $x\in\mathbb{R}$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}} R_n(x,t) K_h(t,x)\,dt\right| &\le C_1 \int_{\mathbb{R}} \left| \int_x^t \frac{(t-u)^{n-1}}{(n-1)!} f^{(n)}(u)\,du \right| h^{-1}\left(h^{-1}|t-x|+1\right)^{-\alpha} dt \\ &\le C_2 \left\|f^{(n)}\right\|_\infty \int_{\mathbb{R}} |t-x|^n h^{-1}\left(h^{-1}|t-x|+1\right)^{-\alpha} dt \\ &\le C_3 \left\|f^{(n)}\right\|_\infty h^{\alpha-1} \int_0^\infty \frac{|y|^n}{|y+h|^\alpha}\,dy \le C_4 h^n \left\|f^{(n)}\right\|_\infty .\end{aligned}$$

◆

We can now show a Strang-Fix type result that will become useful in Section 6.5. It is a simpler version of Theorem 6.5, but handles the case of "simultaneous" approximation from a sequence of PSI spaces.

Theorem 6.10 Let $\{\rho_j\}_{j\geq 1}$ be a sequence of measurable univariate functions and let $m \geq 1$. Assume the following conditions hold for each $j \geq 1$:

1. (uniformly bounded support) $supp(\rho_j) \subseteq [-L, L]$,
2. (uniformly bounded norm) $\|\rho_j\|_\infty \leq M$,
3. (Strang-Fix) $\hat{\rho}_j(0) = 1$, $\hat{\rho}_j^{(l)}(2\pi k) = 0$, $l = 0,\ldots,m-1$, $k \neq 0$,
4. (Poisson Summation) ρ_j satisfies the Poisson summation condition of order m.

Then, there exist constants $\tilde{C}_1, \tilde{C}_2$ which depend only on L, M, m such that:

(i) For any $f \in W_p^m(\mathbb{R})$

$$E\left(f, S(\rho_j)^h\right)_p \leq \tilde{C}_1 h^m \left\|f^{(m)}\right\|_p, \qquad j \geq 1. \tag{6.18}$$

(ii) For any $f \in L_p(\mathbb{R})$

$$E\left(f, S(\rho_j)^h\right)_p \leq \tilde{C}_2 \omega_m(f, h)_p, \qquad j \geq 1. \tag{6.19}$$

Proof

1. From conditions 1,2 it is obvious that the moments of $\{\rho_j\}$ are uniformly bounded. Namely,

$$\left|\hat{\rho}_j^{(l)}(0)\right| = \left|\widehat{(\cdot)^l \rho_j}(0)\right| = \left|\int_{-\infty}^{\infty} x^l \rho_j(x)\,dx\right| = \left|\int_{-L}^{L} x^l \rho_j(x)\,dx\right| \leq M \left\|x^l\right\|_{L_1([L,L])}, \quad j \geq 1,\ l = 1,\ldots,m-1.$$

2. From the discussion that follows Lemma 6.7, we see that conditions 1,2 together with the uniform bounded moments property ensure uniform bound on the kernels $K_{h,j} := K_h(\rho_j)$ of the operators $Q_{h,j} := Q_h(\rho_j)$ defined by (6.13)

$$\left|K_{h,j}(t,x)\right| \begin{cases} = 0 & |x-t| \geq 2Lh, \\ \leq 2LCMh^{-1} & (t,x) \in \mathbb{R}^2. \end{cases}$$

3. Let $f \in W_p^m(\mathbb{R})$. Since for any $j \geq 1$, ρ_j satisfies the SF and Poisson conditions, the kernel $K_{h,j}$ reproduces polynomials. Consequently, for any $x \in \mathbb{R}$

$$\left|f(x) - Q_{h,j}(f,x)\right| = \left|Q_{h,j}(f - T_{m-1}, x)\right| = \left|\int_{\mathbb{R}} R_m(x,t) K_{h,j}(t,x)\,dt\right|,$$

where $T_{m-1}(x,t)$ is the Taylor expansion of degree $m-1$ about the point x. We obtain (6.18) using Lemma 6.9

$$E\left(f,S(\rho_j)^h\right)_p \le \|f-Q_{h,j}f\|_p = \left\|\int_{\mathbb{R}} R_m(x,t)K_{h,j}(t,x)\,dt\right\|_p \le \tilde{C}_1 h^m \left\|f^{(m)}\right\|_p.$$

4. To obtain the estimate (6.19) we observe that the quasi-interpolation operators $Q_{h,j}$ defined by (6.12) for each ρ_j are uniformly bounded for all h,j,p. Denoting by $\tilde{M}$ this uniform bound and using the estimate (6.18) we can apply Theorem 5.2 of chapter 7 in [DL] to derive (6.19). Also, it is also shown in [DL] that the constant $\tilde{C}_2$ depends only on $\tilde{M},m$.

◆

Motivated by applications and following Sweldens and Piessens ([SP]), Unser ([U]) performed a "fine" analysis of certain constants associated to the approximation power of generators. The main difference is that Sweldens and Piessens produced pointwise estimates using wavelets, while Unser's analysis for the global case $p=2$ was done using scaling functions. Unser justified his approach by the fact that wavelets inherit their approximation properties from the scaling functions. Later on in [BU1] Blu and Unser corrected and expanded the approach of [U] replacing the time domain analysis with Fourier techniques. Our motivation to dwell on these "finer" analysis issues comes from the fact that we have constructed non-stationary wavelets for non-refinable generators, that by this "fine" analysis, perform well and in some cases are optimal. Here, still using time domain methods we generalize [BU1] and treat the full range $1\le p\le\infty$. However, in the following we continue to restrict ourselves to compactly supported quasi-interpolation kernels for $p\ne\infty$. We shall later see that for $p=2$, the results of [BU1] are more general, since globally supported quasi-interpolation approximations are treated. First we require the following lemma.

Lemma 6.11 For any $f\in W_p^1(\mathbb{R})$, $1\le p<\infty$ and $h>0$ we have the following "numerical integration" inequality

$$\left\|h\sum_k |f(x+kh)|^p\right\|_{L_\infty(\mathbb{R})}^{1/p} \le \|f\|_{L_p(\mathbb{R})} + h\|f'\|_{L_p(\mathbb{R})}. \tag{6.20}$$

Proof We fix $x\in\mathbb{R}$ and define the following step function

$$\phi_{x,h}(t) := \sum_{k\in\mathbf{Z}} f(x+kh)\,\chi_{[x+kh,x+(k+1)h)}(t).$$

Then,

$$\begin{aligned}\sum_{k\in\mathbf{Z}} h|f(x+kh)|^p &= \sum_{k\in\mathbf{Z}}\int_0^h |f(x+kh)|^p\,dt\\ &\le \sum_{k\in\mathbf{Z}}\int_0^h \left(|f(x+kh+t)| + |f(x+kh+t)-f(x+kh)|\right)^p dt\end{aligned}$$

$$= \left\| |f| + |f - \phi_{x,h}| \right\|_p^p$$
$$\leq \left(\|f\|_p + \|f - \phi_{x,h}\|_p \right)^p.$$

Therefore, it is sufficient to prove that $\|f - \phi_{x,h}\|_p \leq h\|f'\|_p$. Let $t \in [x + kh, x + (k+1)h)$ for some $k \in \mathbb{Z}$. Then using the Hölder inequality,

$$\begin{aligned}
|f(t) - \phi_{x,h}(t)|^p &= |f(t) - f(x+kh)|^p \\
&= \left| \int_{x+kh}^{x+kh+t} f'(u)\,du \right|^p \\
&\leq \left| \int_{x+kh}^{x+(k+1)h} |f'(u)|\,du \right|^p \\
&\leq h^{p-1} \|f'\|_{L_p([x+kh,x+(k+1)h])}^p .
\end{aligned}$$

We now apply the p norm to the last inequality

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(t) - \phi_{x,h}(t)|^p\,dt &= \sum_k \int_{x+kh}^{x+(k+1)h} |f(t) - \phi_{x,h}(t)|^p\,dt \\
&\leq h^{p-1} \sum_k \int_{x+kh}^{x+(k+1)h} \left(\int_{x+kh}^{x+(k+1)h} |f'(u)|^p\,du \right) dt \\
&= h^p \sum_k \int_{x+kh}^{x+(k+1)h} |f'(u)|^p\,du \\
&= h^p \|f'\|_p^p .
\end{aligned}$$

♦

Theorem 6.12 Let $1 \leq p < \infty$. Assume $\phi \in L_\infty(\mathbb{R})$ has compact support and provides L_p approximation order m using polynomial reproducing kernels $K_h(\phi)$ of type (6.13). Then,

1. for any function $f \in W_p^{m+1}(\mathbb{R})$

$$E\left(f, S(\phi)^h\right)_p \leq C_{p,K}^- h^m \left\|f^{(m)}\right\|_p + C h^{m+1} \left\|f^{(m+1)}\right\|_p, \tag{6.21}$$

where

$$C_{p,K}^- := \frac{1}{m!} \|e_{m,K}(x)\|_{L_p([0,1])}, \quad e_{m,K}(x) := (-1)^{m-1}\left(x^m - \int_{\mathbb{R}} t^m K(t,x)\,dt\right). \tag{6.22}$$

2. We can bound the constants $C^{-}_{p,K}$ by

$$C^{-}_{p,K} \le C^{-}_{\infty,K} \le \frac{2^{-2m+2}}{m!}.$$

Proof Let $f \in W_p^{m+1}(\mathbb{R})$. Then for a fixed $x \in \mathbb{R}$,

$$\begin{aligned} f(x)-Q_h(f,x) &= Q_h\left(f-T_{m-1}(f,x)\right) = \int_{\mathbb{R}} R_m(x,t)\frac{1}{h}K\left(\frac{t}{h},\frac{x}{h}\right)dt \\ &= \int_{\mathbb{R}}\left(\frac{f^{(m)}(x)(x-t)^m}{m!}+R_{m+1}(x,t)\right)\frac{1}{h}K\left(\frac{t}{h},\frac{x}{h}\right)dt \\ &= \frac{f^{(m)}(x)h^m}{m!}\int_{\mathbb{R}}\left(\frac{x}{h}-t\right)^m K\left(t,\frac{x}{h}\right)dt + \int_{\mathbb{R}} R_{m+1}(x,t)K_h(t,x)dt. \end{aligned}$$

Next, following [U], we show the equivalence

$$e_{m,K}(x) = \int_{\mathbb{R}}(x-t)^m K(t,x)\,dt. \tag{6.23}$$

Indeed, since the kernel K reproduces polynomials of degree $< m$ we have that

$$\begin{aligned} \int_{\mathbb{R}}(x-t)^m K(t,x)\,dt &= \sum_{k=0}^{m-1}(-1)^k\binom{m}{k}x^{m-k}\int_{\mathbb{R}} t^k K(t,x)\,dt + (-1)^m\int_{\mathbb{R}} t^m K(t,x)\,dt \\ &= x^m\left(\sum_{k=0}^{m-1}(-1)^k\binom{m}{k}\right)+(-1)^m\int_{\mathbb{R}} t^m K(t,x)\,dt \\ &= x^m\left(\sum_{k=0}^{m}(-1)^k\binom{m}{k}\right)-(-1)^m x^m+(-1)^m\int_{\mathbb{R}} t^m K(t,x)\,dt \\ &= (-1)^m\left(\int_{\mathbb{R}} t^m K(t,x)\,dt - x^m\right) = e_{m,K}(x). \end{aligned}$$

Thus pointwise

$$f(x)-Q_h(f,x) = \frac{f^{(m)}(x)h^m}{m!}e_{m,K}\left(\frac{x}{h}\right)+\int_{\mathbb{R}} R_{m+1}(x,t)K_h(t,x)dt,$$

where $e_{m,K}$ is the $m-$th order "error moment" of the kernel defined by (6.22). We obtain a bound with two terms

$$\left\|f(x)-Q_h(f,x)\right\|_p \le \frac{h^m}{m!}\left\|f^{(m)}(x)e_m\left(\frac{x}{h}\right)\right\|_p + \left\|\int_{\mathbb{R}} R_{m+1}(x,t)K_h(t,x)dt\right\|_p. \tag{6.24}$$

First we wish to bound the second term in (6.24). If the kernels K_h are bounded using (6.14), then using Lemma 6.9 with the choice $n = m+1$ we have the estimate

$$\left\| \int_{\mathbb{R}} R_{m+1}(x,t) K_h(t,x) dt \right\|_p \le C_1 h^{m+1} \left\| f^{(m+1)} \right\|_p .$$

We can now assume that $1 \le p < \infty$, since for $p = \infty$ estimate (6.21) is immediate from the last inequality and proceed with the first term. Using (6.13) and (6.23), it is easy to verify that $e_{m,K}(x)$ is 1-periodic. Therefore, we derive the following

$$\begin{aligned}
\left\| f^{(m)}(x) e_{m,K}\left(\frac{x}{h}\right) \right\|_p^p &= \sum_k \int_0^h \left| f^{(m)}(x+kh) \right|^p \left| e_{m,K}\left(\frac{x}{h}\right) \right|^p dx \\
&= \int_0^h \left| e_{m,K}\left(\frac{x}{h}\right) \right|^p \sum_k \left| f^{(m)}(x+kh) \right|^p dx \\
&= \int_0^1 \left| e_{m,K}(y) \right|^p h \sum_k \left| f^{(m)}(hy+kh) \right|^p dy \\
&\le \left\| e_{m,K} \right\|_{L_p([0,1])}^p \left\| h \sum_k \left| f^{(m)}(x+kh) \right|^p \right\|_{L_\infty(\mathbb{R})} .
\end{aligned}$$

Since $f \in W_p^{m+1}(\mathbb{R})$ we have that $f^{(m)}$ is absolutely continuous. Using our "numerical integration" inequality (6.20) we obtain

$$\left\| h \sum_k \left| f^{(m)}(x+kh) \right|^p \right\|_{L_\infty(\mathbb{R})}^{1/p} \le \left\| f^{(m)} \right\|_{L_p(\mathbb{R})} + h \left\| f^{(m+1)} \right\|_{L_p(\mathbb{R})} .$$

Thus we can combine the above estimates of the two terms and obtain estimate (6.21) by

$$\begin{aligned}
\left\| f(x) - Q_h(f,x) \right\|_p &\le \frac{h^m}{m!} \left\| e_{m,K} \right\|_{L_p([0,1])} \left(\left\| f^{(m)} \right\|_p + h \left\| f^{(m+1)} \right\|_p \right) + C_1 h^{m+1} \left\| f^{(m+1)} \right\|_p \\
&\le C_{p,K}^- h^m \left\| f^{(m)} \right\|_p + C_2 h^{m+1} \left\| f^{(m+1)} \right\|_p .
\end{aligned}$$

Next we show that $C_{\infty,K}^- \le 2^{-2m+2}/m!$. Using (6.22)

$$m! C_{\infty,K}^- = \left\| e_{m,K} \right\|_{L_\infty([0,1])} = \left\| x^m - \int_{\mathbb{R}} t^m K(t,x) dt \right\|_{L_\infty([0,1])} .$$

Let $A_m(x) := 2^{-2m+1} C_m(2x-1)$ with $C_m(x)$ the Chebyshev polynomial of degree m. Since $\|C_m\|_{L_\infty([-1,1])} = 1$ (see section 3.6 in [DL]) we have $\|A_m(x)\|_{L_\infty([0,1])} = 2^{-2m+1}$. As the leading coefficient of A_m is 1, the polynomial $D_m(x) := x^m - A_m(x)$ is in Π_{m-1} and thus reproduced by the kernel $K(x,t)$. Recalling the "partition of unity" property $\int_{\mathbb{R}} K(t,x)\,dt \equiv 1$ and using the notation $\|\cdot\| := \|\cdot\|_{L_\infty([0,1])}$ we obtain the estimate

$$\begin{aligned}
\left\|x^m - \int_{\mathbb{R}} t^m K(t,x)\,dt\right\| &\le \left\|x^m - D_m(x)\right\| + \left\|\int_{\mathbb{R}} \left(t^m - D_m(t)\right) K(t,x)\,dt\right\| \\
&= \left\|A_m(x)\right\| + \left\|\int_{\mathbb{R}} A_m(t) K(t,x)\,dt\right\| \\
&\le 2^{-2m+1}\left(1 + \left\|\int_{\mathbb{R}} K(t,x)\,dt\right\|\right) = 2^{-2m+2}.
\end{aligned}$$

♦

It is not too surprising to see that the leading constant $C_{p,K}^{-}$ (6.22) is determined by how well the quasi-interpolation kernel approximates the polynomial x^m in the p norm, since it is the polynomial with lowest degree that is not reproduced. Also observe that if ϕ, using the quasi-interpolation kernel K, provides higher approximation order than m, then as expected $C_{p,K}^{-} = 0$.

As Unser pointed out, the constant $C_{p,K}^{-}$ can be used to evaluate the performance of different kernels with the same approximation power. In fact, it would be interesting to find "optimal" kernels that minimize $C_{p,K}^{-}$ for each given p and support size, as was done for $p = 2$ and support size m in [BTU].

6.3 L_2 approximation from shift invariant spaces

In the previous section, the analysis of approximation from shift invariant spaces for $1 \le p \le \infty$ was carried out in the "time domain" using pointwise estimates. For the case of $p = 2$, two tools allow the analysis to be both elegant and powerful, the Hilbert space geometry and the Plancharel-Parseval equality. The latter allows us to carry out the analysis in the "frequency domain". An excellent survey of L_2 approximation from shift invariant spaces is [JP].
In this section, for any $f \in L_2(\mathbb{R}^d)$, $V \subset L_2(\mathbb{R}^d)$ we use the notation

$$E(f,V) := E(f,V)_{L_2(\mathbb{R}^d)}.$$

Applying Fourier methods one can use a certain error kernel to obtain L_2 estimates. Moreover, using the error kernel introduced below one can provide complete characterization of the approximation order of shift invariant spaces.

Definition 6.13 [BDR1] Let $\phi \in L_2(\mathbb{R}^d)$. We define the following error kernel $\Lambda_\phi \in L_\infty\left([-\pi,\pi]^d\right)$

$$\Lambda_\phi := \left(1 - \frac{|\hat{\phi}|^2}{[\hat{\phi},\hat{\phi}]}\right)^{\frac{1}{2}}, \tag{6.25}$$

where 0/0 is interpreted to be 0.

The following theorem characterizes the approximation order of an SI space, by the existence of a **superfunction**. The superfunction is required to have an error kernel (6.25) with a fast decay to zero about the origin.

Theorem 6.14 [BDR3] Let V be an SI space. Then V provides approximation order $m \in \mathbb{R}_+$ such that

$$E(f,V^h) \le C_V h^m \|f\|_{H^m}.$$

if and only if there exists $\phi \in V$ for which $|\cdot|^{-m} \Lambda_\phi \in L_\infty(B)$, where B is some neighborhood of the origin.

This result should be compared with the Strang-Fix conditions (6.3) and Theorem 6.5. Assume $V = S(\phi)$, where ϕ is compactly supported and satisfies the SF conditions of order $m \in \mathbb{N}$. Then, from the SF conditions we have that $\hat{\phi}(0) \ne 0$, $\hat{\phi}(2\pi k) = 0$ for all $0 \ne k \in \mathbb{Z}^d$

and so

$$\Lambda_\phi^2(0)=1-\frac{|\hat{\phi}(0)|^2}{[\hat{\phi},\hat{\phi}](0)}=1-\frac{|\hat{\phi}(0)|^2}{|\hat{\phi}(0)|^2}=0,$$

Also, as we assumed that ϕ has compact support, $[\hat{\phi},\hat{\phi}]$ is a polynomial. It can be shown that in such a case $w^{-1}\Lambda_\phi(w)$ is in $L_\infty([-\pi,\pi]^d)$ and using the higher order SF conditions, that $|\cdot|^{-m}\Lambda_\phi \in L_\infty([-\pi,\pi]^d)$. Therefore the SF conditions are (as they must be) a special case of the characterization of Theorem 6.15.

Also, we note the following:

1. The characterization of Theorem 6.14 treats the case of general order $m\in\mathbb{R}_+$.
2. There are no restrictions on the support of the generators of V (they need not even decay at infinity).
3. Perhaps the most interesting observation is that Theorem 6.14 also handles a case not covered by the SF conditions where $\hat{\phi}$ vanishes at the origin.
4. The above characterization uses the Sobolev norm in the estimate (6.1) rather than the semi-norm.

The following result generalizes this characterization to the non-stationary case. It is interesting to compare it to the "simultaneous" approximation result of Theorem 6.10.

Theorem 6.15 [BDR3] A non-stationary ladder $V^h=S(\phi_h)^h$ provides approximation order m if and only if for some h_0 the functions $(h+|\cdot|)^{-m}\Lambda_{\phi_h}$, $h<h_0$ are uniformly bounded in $L_\infty(B)$, where B is some neighborhood of the origin.

The error kernel (6.25) can be used not only for characterization of the approximation order of SI spaces but also for "finer" error estimates of type (6.21) for the case $p=2$. Recall that in Section 6.2 we derived estimates for the general case of $1\le p\le\infty$, but only the case of local shift invariant spaces was treated. Furthermore, we restricted our estimates to compactly supported quasi-interpolation kernels of type (6.13). For $p=2$ the orthogonal projection operator is typically not of local nature even if the shift invariant space is local. This can be seen using (2.13) for the PSI case or (2.14) for the general FSI case. There is a certain technical difficulty to perform analysis of such globally supported approximation kernels in the time-domain and this difficulty can be removed in the frequency domain. As proved in [BU1] the kernel (6.25) produces very accurate error estimates.

Theorem 6.16 [BU1] Let $\phi \in E_m(\mathbb{R}^d)$ be stable. For all $f \in H^r$ with $r > 1/2$ we have

$$E\left(f, S(\phi)^h\right) = \left[\int \left|\hat{f}(w)\right|^2 \Lambda_\phi^2(hw)\,dw\right]^{1/2} + e(f,h),$$

with

$$\left|e(f,h)\right| \le \gamma_\phi h^r |f|_{H^r}, \quad \gamma_\phi := \frac{2}{\pi^r}\sqrt{\zeta(2r)\left\|\Lambda_\phi^2\right\|_\infty}, \quad \zeta(s) := \sum_{n\ge1}\frac{1}{n^s}.$$

Theorem 6.17 [BU1] Assume that $\phi \in E_m(\mathbb{R}^d)$ is stable and provides L_2 approximation order m. For all $f \in H^r$ with $r \ge m$ we have

$$E\left(f, S(\phi)^h\right) \le C_\phi^- h^m |f|_{H^m} + \gamma_\phi h^r |f|_{H^r},$$

where

$$C_\phi^- := \sqrt{\frac{\left\|\left(\Lambda_\phi^2\right)^{(2m)}\right\|_\infty}{(2m)!}}.$$

Theorem 6.18 [BU1] Assume that $\phi \in E_m(\mathbb{R}^d)$ is stable with $\hat{\phi}(0) = 1$ and provides L_2 approximation order $m \in \mathbb{N}$. Then for any function $f \in H^{m+1}(\mathbb{R}^d)$

$$E\left(f, S(\phi)^h\right) = C_\phi^- h^m |f|_{H^m(\mathbb{R}^d)} + O\left(h^{m+1}\right), \quad \text{where } C_\phi^- = \frac{1}{m!}\sqrt{\sum_{k\ne0}\left|\hat{\phi}^{(m)}(2\pi k)\right|^2}. \tag{6.26}$$

It is interesting to compare this last result with the estimate (6.21). Formally, neglecting the technical difficulties of globally supported kernels, we can identify the constants $C_\phi^- = C_{2,K}^-$ appearing in (6.21), (6.26) where K is the kernel corresponding to the orthogonal projection

$$K(t,x) = \sum_k \tilde{\phi}(t-k)\phi(x-k),$$

with $\tilde{\phi}$ the "natural" dual of ϕ. Indeed in such a case, there exists a formal explanation ([U]) of the equivalence of the constants $C_{2,K}^-, C_\phi^-$, the first obtained in the time-domain and the second in the frequency domain.

One of the results in [U] is that the leading constants of type C_ϕ^- in (6.26) are much smaller for the B-Spline generators than for the Daubechies orthonormal scaling functions. Since wavelets directly inherit this constant from the scaling functions, it might explain the empirical evidence that spline wavelets outperform in image coding the Daubechies wavelets that possess the same number of vanishing moments. In [BTU] the authors went a step further and constructed the O-Moms generators (see Example 6.6) which for a given approximation order and smallest possible support size, have a minimal constant C_ϕ^-. Indeed, a surprising result is that smoothness is not essential for securing approximation order.

Example 6.19 [BTU] The optimal generator with approximation order four and minimal compact support is $OM_4 := N_4 + N_4''/42$ where N_4 is the fourth order B-spline. The (normalized) gain in sampling density brought by using OM_4 instead of N_4 is

$$\left(\frac{C_{N_4}^-}{C_{OM_4}^-}\right)^{\frac{1}{4}} \approx 1.436.$$

♦

As discussed in Chapter 3, the generator OM_4 is not refinable in the classical sense. Nevertheless, as shown in Example 3.7 it is three-scale refinable. Thus, we can see that by relaxing the two-scale refinability constraint, one can find good generators, which in some sense are optimal.

Our next goal is to add to the L_2 – superfunction theory a more careful treatment of constants. We require the following two lemmas.

Lemma 6.20 [BDR1] Let V be an SI space. Then for any $f, g \in L_2(\mathbb{R}^d)$

$$E(f,V) \le E(f,S(P_V g)) \le E(f,V) + 2E(f,S(g)).$$

Next we consider the well-known sinc-function g^* defined by

$$g^* := \prod_{i=1}^{d} \frac{\sin \pi x_i}{\pi x_i}, \qquad \widehat{g^*} = \chi_{[-\pi,\pi]^d}. \tag{6.27}$$

It is an ideal superfunction, since it has "infinite" approximation order.

Lemma 6.21 The PSI space generated by the sinc-function defined by (6.27) has approximation power r for any $r \in \mathbb{R}_+$. Namely, for any $f \in H^r(\mathbb{R}^d)$, $r > 0$ and $h > 0$ one has

1. The "norm" estimate ([BDR1])

$$E\left(f, S\left(g^*\right)^h\right) \le \varepsilon_{f,r}(h) h^r \|f\|_{H^r}, \qquad \varepsilon_{f,r}(h)^2 := \frac{\int_{(\mathbb{R}^d \setminus T^d)/h} \left(1+|\cdot|\right)^{2r} |\hat{f}|^2}{\int_{\mathbb{R}^d} \left(1+|\cdot|\right)^{2r} |\hat{f}|^2}. \tag{6.28}$$

2. The "semi-norm" estimate (see for example [JP])

$$E\left(f, S\left(g^*\right)^h\right) \le h^r |f|_{H^r}. \tag{6.29}$$

It is interesting to compare the two estimates. Although the first estimate uses the norm of the given function, it shows that the sinc-function provides infinite density order. Namely, since $\varepsilon_{f,r}(h)$, defined in (6.28) approaches zero as h tends to zero, we have that

$$E\left(f, S\left(g^*\right)^h\right) = o\left(h^r\right), \qquad h \to 0.$$

On the other hand estimate (6.29) is a standard Jackson type estimate using the semi-norm.

We now combine the finer error estimates of [BU1] related to optimal constants with the superfunction theory of [BDR1]. We show that the superfunction provides asymptotically exactly the same approximation as the "full" space, with the same (sharp) leading constant.

Theorem 6.22 Let V be an FSI space with approximation order $m \in \mathbb{R}_+$, such that for any function $f \in H^r\left(\mathbb{R}^d\right)$, $r \geq m$

$$E\left(f, V^h\right) \leq C_V^- h^m |f|_{H^m} + O\left(h^r\right). \tag{6.30}$$

Then there exists a superfunction $\phi \in V$ such that for any $f \in H^r\left(\mathbb{R}^d\right)$, $r \geq m$ one has

$$E\left(f, S(\phi)^h\right) \leq C_V^- h^m |f|_{H^m} + O\left(h^r\right).$$

Proof Let $f \in H^r\left(\mathbb{R}^d\right)$. We use a dilated version of (6.30)

$$E\left(f(h\cdot), V\right) = h^{-d/2} E\left(f, V^h\right) \leq h^{-d/2}\left(C_V^- h^m |f|_{H^m} + C(V,r,f) h^r\right).$$

Select $\phi = P_V g^*$, where g^* is the multivariate sinc-function (6.27). We apply Lemma 6.20 and use the estimates (6.29), (6.30) to derive

$$\begin{aligned}
E\left(f, S(\phi)^h\right) &= h^{d/2} E\left(f(h\cdot), S(\phi)\right) \\
&\leq h^{d/2}\left[E\left(f(h\cdot), V\right) + 2E\left(f(h\cdot), S\left(g^*\right)\right)\right] \\
&\leq C_V^- h^m |f|_{H^r} + C(V,r,f) h^r + 2h^r |f|_{H^r} \\
&\leq C_V^- h^m |f|_{H^r} + O\left(h^r\right).
\end{aligned}$$

♦

We see that the projection of the ideal sinc-function to the FSI space not only provides a characterization of the approximation order of the space, but also satisfies any fine estimates that hold for the FSI space.

To show a similar result for local shift invariant spaces (see Definition 2.26), we first require the following "superfunction" result.

Theorem 6.23 [BDR2] Let V be a local FSI space. Let g be any compactly supported function (not necessarily in V). Then, there exists a compactly supported function $\phi \in V$ such that, for every $f \in L_2(\mathbb{R}^d)$

$$E(f,S(\phi)) \le E(f,V) + 2E(f,S(g)). \tag{6.31}$$

Theorem 6.24 If in addition to the assumptions of Theorem 6.22 we further assume that V is local, then for any fixed $r \in \mathbb{N}$ with $r > m$ there exists a compactly supported function $\phi_r \in V$ such that for any $f \in H^r(\mathbb{R}^d)$ one has

$$E\left(f,S(\phi_r)^h\right) \le C_V^- h^m |f|_{H^m} + O(h^r).$$

Proof The proof is a straight forward application of Theorem 6.23 with the selection $g = N_r$, where N_r is the tensor-product B-spline of order r. It is well known that for each function $f \in H^r(\mathbb{R}^d)$

$$E\left(f,S(N_r)^h\right) \le C_r h^r |f|_{H^r}.$$

Thus, there exists a compactly supported $\phi_r \in V$ that satisfies (6.31). Using the dilation equality $E\left(f,S(\phi_r)^h\right) = h^{d/2}E(f(h\cdot),S(\phi_r))$ and applying the method of proof of Theorem 6.22, we can use (6.31) to estimate,

$$\begin{aligned} E\left(f,S(\phi_r)^h\right) &\le C_V^- h^m |f|_{H^m} + C(V,r,f)h^r + 2C_r h^r |f|_{H^r} \\ &\le C_V^- h^m |f|_{H^m} + O(h^r). \end{aligned}$$

◆

6.4 Approximation properties of the non-stationary Superfunction wavelets

We now go back to the superfunction decompositions of Section 5.1 and verify that the non-stationary half-multiresolution inherits the approximation properties of the initial space and the reference space. First, we need the following result.

Theorem 6.25 Let $\rho_0,\phi \in L_2(\mathbb{R}^d)$ have approximation order m and assume $S(\rho_0)^{1/2} = S(\rho_1) \oplus S(\Psi)$ where $S(\Psi) \perp S(\phi)$. Then ρ_1 has approximation order m. Furthermore:

1. If for all functions $f \in H^m(\mathbb{R}^d)$ and $h>0$ the following two estimates hold

$$E\left(f,S(\rho_0)^h\right) \le C_{\rho_0} h^m |f|_{H^m(\mathbb{R}^d)}, \qquad E\left(f,S(\phi)^h\right) \le C_\phi h^m |f|_{H^m(\mathbb{R}^d)}, \tag{6.32}$$

then for all functions $f \in H^m(\mathbb{R}^d)$ and $h>0$

$$E\left(f,S(\rho_1)^h\right) \le C_{\rho_1} h^m |f|_{H^m(\mathbb{R}^d)}, \qquad C_{\rho_1} \le C_{\rho_0} 2^{-m} + 2C_\phi .$$

2. If for all functions $f \in H^r(\mathbb{R}^d)$, $r>m$ and $h>0$ the following two estimates hold

$$E\left(f,S(\rho_0)^h\right) \le C_{\rho_0}^- h^m |f|_{H^m(\mathbb{R}^d)} + O(h^r), \qquad E\left(f,S(\phi)^h\right) \le C_\phi^- h^m |f|_{H^m(\mathbb{R}^d)} + O(h^r), \tag{6.33}$$

then for all functions $f \in H^r(\mathbb{R}^d)$ and $h>0$

$$E\left(f,S(\rho_1)^h\right) \le C_{\rho_1}^- h^m |f|_{H^m(\mathbb{R}^d)} + O(h^r), \qquad C_{\rho_1}^- \le C_{\rho_0}^- 2^{-m} + 2C_\phi^- .$$

Proof

1. Let $f \in H^m(\mathbb{R}^d)$. Since ρ_0,ϕ have approximation power m, we can obtain a dilated version of (6.2) for both generators

$$E\left(f(h\cdot),S(\rho_0)^{1/2}\right) \le h^{-d/2} C_{\rho_0} (h/2)^m |f|_{H^m(\mathbb{R}^d)}, \qquad E\left(f(h\cdot),S(\phi)\right) \le h^{-d/2} C_\phi h^m |f|_{H^m(\mathbb{R}^d)}.$$

Since $S(\Psi) \perp S(\phi)$ we have that $P_{S(\rho_0)^{1/2}} S(\phi) \subseteq S(\rho_1)$. Recall from Theorem 2.4 that the shift and projection operators commute. Thus,

$$P_{S(\rho_0)^{1/2}} S(\phi) = S\left(P_{S(\rho_0)^{1/2}} \phi\right).$$

We now apply Lemma 6.20 to derive the following inequality

$$\begin{aligned}
E\left(f, S(\rho_1)^h\right) &= h^{d/2} E\left(f(h\cdot), S(\rho_1)\right) \\
&\le h^{d/2} E\left(f(h\cdot), S\left(P_{S(\rho_0)^{1/2}}\phi\right)\right) \\
&\le h^{d/2}\left[E\left(f(h\cdot), S(\rho_0)^{1/2}\right) + 2E\left(f(h\cdot), S(\phi)\right)\right] \\
&\le \left(C_{\rho_0} 2^{-m} + 2C_\phi\right) h^m |f|_{H^m}.
\end{aligned}$$

2. Let $f \in H^r\left(\mathbb{R}^d\right)$. Then by the same arguments

$$\begin{aligned}
E\left(f, S(\rho_1)^h\right) &\le h^{d/2}\left[E\left(f(h\cdot), S(\rho_0)^{1/2}\right) + 2E\left(f(h\cdot), S(\phi)\right)\right] \\
&\le \left(C_{\rho_0}^{-} 2^{-m} + 2C_\phi^{-}\right) h^m |f|_{H^m} + \left(C(\rho_0, f) 2^{-r} + 2C(\phi, f)\right) h^r.
\end{aligned}$$

♦

We are now ready to justify the superfunction construction of Theorem 5.6.

Corollary 6.26 Let $\phi, \rho_0 \in L_2\left(\mathbb{R}^d\right)$ have approximation order m. Let $\{\rho_j\}_{j\ge 1}$ be such that for all $j \ge 1$:

a. $S(\rho_j) \oplus S(\Psi_j) = S(\rho_{j-1})^{1/2}$,

b. $S(\Psi_j) \perp S(\phi)$.

1. If ϕ, ρ_0 satisfy (6.32), then we also have the uniform estimate for any $f \in H^m\left(\mathbb{R}^d\right)$

$$E\left(f, S(\rho_j)^h\right) \le \frac{2^{m+1}}{2^m - 1} \max\left(C_{\rho_0}, C_\phi\right) h^m |f|_{H^m(\mathbb{R}^d)}, \quad j \ge 0. \tag{6.34}$$

2. If ϕ, ρ_0 satisfy (6.33), then we also have the uniform estimate for any $f \in H^r\left(\mathbb{R}^d\right)$, $r > m$

$$E\left(f, S(\rho_j)^h\right) \le \frac{2^{m+1}}{2^m - 1} \max\left(C_{\rho_0}^{-}, C_\phi^{-}\right) h^m |f|_{H^m(\mathbb{R}^d)} + O\left(h^r\right), \quad j \ge 0. \tag{6.35}$$

Proof The proof is by induction. We only prove (6.34) because the proof for (6.35) is similar. The estimate (6.34) is certainly true for the initial function ρ_0. Assume that ρ_{j-1} has approximation power m. By Theorem 6.25 we see that the generator ρ_j, constructed using the projection method of Theorem 5.6, inherits the approximation power m with a constant $C_{\rho_j} \le 2^{-m} C_{\rho_{j-1}} + 2C_\phi$. The relation leads to the uniform bound

$$C_{\rho_j} \le 2^{-jm} C_{\rho_0} + \left(\sum_{n=0}^{j-1} 2^{1-nm} \right) C_\phi$$
$$\le \left(2^{-jm} + \sum_{n=0}^{j-1} 2^{1-nm} \right) \max\left(C_{\rho_0}, C_\phi\right)$$
$$\le \frac{2^{m+1}}{2^m - 1} \max\left(C_{\rho_0}, C_\phi\right).$$

♦

Example 6.27 Select ϕ, ρ_0 in Corollary 6.26 to be $OM_4 := N_4 + N_4''/42$ (see Example 6.6). Then for any $f \in H^r(\mathbb{R}^d)$, $r > 4$

$$E\left(f, S(\rho_j)^h\right) \le \frac{2^5}{2^4 - 1} C^-_{OM_4} h^4 |f|_{H^4(\mathbb{R}^d)} + O(h^r), \qquad j \ge 0.$$

Therefore for all $j \ge 0$,

$$\left(\frac{C^-_{N_4}}{C^-_{OM_4}} \right)^{\frac{1}{4}} \ge 1.45 \Rightarrow \left(\frac{C^-_{N_4}}{C^-_{\rho_j}} \right)^{\frac{1}{4}} \ge 1.1882.$$

Assume $\{\psi_j\}_{j\ge1}$ are any non-stationary (compactly supported) wavelets, complementing the half-multiresolution generated by $\{\rho_j\}_{j\ge0}$ where OM_4 generates both the initial space and the reference space. Then, these wavelets have a sharp constant smaller then the B-wavelets of [Ch] with a gain of nearly 20%. This result is not very surprising. We have shown (Example 5.10) that we can choose the first wavelet ψ_1 such that $|\text{supp}(\psi_1)| = 7$, which is exactly the support size of the cubic B-wavelet. But as explained in Section 5.1, for any such non-stationary wavelet sequence, the support of the wavelets in general grows.

♦

Here is an alternative proof for the univariate case of Theorem 6.25 that works directly with SF conditions. It generalizes a known result that says that a two-scale refinable scaling function has approximation order m if and only if a complementary wavelet has m moments. As we shall see the method of proof leads to a relation for the associated sharp constants.

Theorem 6.28 Let $\varphi, \phi \in \mathbb{E}_m(\mathbb{R})$ satisfy the SF conditions of order m such that φ is also stable. If $S(\varphi)^{1/2} = S(\rho) \oplus S(\psi)$ is a stable decomposition with the following properties

1. $S(\psi) \perp S(\phi)$,
2. $\rho, \psi \in \mathbb{E}_m(\mathbb{R})$,
3. $P, Q \in \mathbb{W}$, where P, Q are the two-scale symbols of ρ, ψ (see Definition 5.1).

Then,

1. ψ has m vanishing moments. Namely, $\int_{\mathbb{R}} x^l \psi(x)dx = 0$, $l = 0,\ldots,m-1$,
2. ρ satisfies the SF conditions of order m.

Proof Recall from Theorem 5.5 that a necessary condition for the stable semi-orthogonal decomposition $S(\varphi)^{1/2} = S(\rho) \oplus S(\psi)$ is that ψ has a two-scale relation

$$Q(w) = e^{iw} G(w+\pi) K(2w), \qquad 0 \neq K \in W, \tag{6.36}$$

where G is defined by (5.11). We begin by showing that Q has a zero multiplicity m at the origin. Since $\psi \in E_m(\mathbb{R})$, $S(\psi) \perp S(\phi)$ and the shifts of ϕ reproduce polynomials of degree $m-1$, it is easy to show that ψ has m vanishing moments. Thus, $\hat{\psi}^{(l)}(0) = 0$ for $l = 0,\ldots,m-1$. From the two-scale relation

$$\hat{\psi}(w) = Q\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right), \tag{6.37}$$

we have that $0 = \hat{\psi}(0) = Q(0)\hat{\varphi}(0)$. Since for φ the SF conditions hold, $\hat{\varphi}(0) \neq 0$ and so we must have $Q(0) = 0$. Using (6.37) it is easy to see that since $\hat{\varphi}(0) \neq 0$, Q is $m-1$ differentiable at the origin. Assume by induction that $Q^{(k)}(0) = 0$ for $0 \le k < l$. Then

$$0 = \hat{\psi}^{(l)}(0) = 2^{-l} \sum_{k=0}^{l} \binom{l}{k} Q^{(l-k)}(0) \hat{\varphi}^{(k)}(0) = Q^{(l)}(0)\hat{\varphi}(0),$$

so that $Q^{(l)} = 0$. By (6.36) the symbol G has a zero multiplicity m at the point $w = \pi$. Using (5.10), (5.11) this implies that P also has zero multiplicity m at the point $w = \pi$. Next we use the two-scale relation

$$\hat{\rho}(w) = P\left(\frac{w}{2}\right)\hat{\varphi}\left(\frac{w}{2}\right),$$

to show that ρ fulfils the Strang-Fix conditions (6.3):

1. Since $S(\varphi)^{1/2} = S(\rho) \oplus S(\psi)$, there exist 2π-periodic functions m_0, m_1 such that

$$\frac{1}{2}\hat{\varphi}\left(\frac{w}{2}\right) = m_0(w)\hat{\rho}(w) + m_1(w)\hat{\psi}(w).$$

Since $\hat{\varphi}(0) \neq 0$ and $\hat{\psi}(0) = 0$, we conclude that $\hat{\rho}(0) \neq 0$.

2. Let $n \neq 0$. We observe two cases:

The case $n \equiv 0 (\mathrm{mod}\, 2)$: since for φ the SF conditions of order m hold, we have

$$\hat{\rho}(2\pi n) = P(0)\hat{\varphi}(2\pi n') = 0, \qquad 0 \neq n' \in \mathbb{Z}.$$

The case $n \equiv 1 (\mathrm{mod}\, 2)$: since P has a zero at π we have

$$\hat{\rho}(2\pi n) = P(\pi)\hat{\varphi}(\pi n) = 0.$$

Using induction and repeated application of the above argument we can obtain

$$\hat{\rho}^{(l)}(2\pi n) = 0, \qquad 0 \le l \le m-1, \quad n \neq 0,$$

thereby proving that ρ satisfies the SF conditions.

The method of proof in Theorem 6.28 also provides a way to calculate the sharp approximation constant C_ρ^- defined by (6.26). We can obtain for each $0 \neq n \in \mathbb{Z}$

$$\hat{\rho}^{(m)}(2\pi n) = \left(P\left(\frac{w}{2}\right)\varphi\left(\frac{w}{2}\right)\right)^{(m)}\Bigg|_{w=2\pi n} = 2^{-m}\sum_{k=0}^{m}\binom{m}{k} P^{(k)}(\pi n)\hat{\varphi}^{(m-k)}(\pi n).$$

Again there are two cases:

1. The case $n \equiv 0 (\mathrm{mod}\, 2)$
$$\hat{\rho}^{(m)}(2\pi n) = 2^{-m}P(0)\hat{\varphi}^{(m)}(2\pi n'), \qquad n' = n/2.$$
2. The case $n \equiv 1 (\mathrm{mod}\, 2)$
$$\hat{\rho}^{(m)}(2\pi n) = 2^{-m}P^{(m)}(\pi)\hat{\varphi}(\pi n).$$

Using formula (6.26) for the sharp constant, we can merge the two cases and obtain

$$\begin{aligned}
\left(C_\rho^-\right)^2 &= \frac{1}{(m!)^2}\sum_{n\neq 0}\left|\hat{\rho}^{(m)}(2\pi n)\right|^2 \\
&= \frac{1}{(m!)^2}\left[\sum_{k\neq 0}\left|\hat{\rho}^{(m)}(2\pi 2k)\right|^2 + \sum_{k}\left|\hat{\rho}^{(m)}(2\pi(2k+1))\right|^2\right] \\
&= \frac{1}{(m!)^2}\left[\sum_{k\neq 0}\left|2^{-m}P(0)\hat{\varphi}^{(m)}(2\pi k)\right|^2 + \sum_{k}\left|2^{-m}P^{(m)}(\pi)\hat{\varphi}(\pi(2k+1))\right|^2\right]
\end{aligned}$$

$$= \frac{1}{(m!)^2 2^{2m}} \left[(m!)^2 |P(0)|^2 (C_\varphi^-)^2 + |P^{(m)}(\pi)|^2 [\hat{\varphi}, \hat{\varphi}](\pi) \right]. \tag{6.38}$$

To see how (6.38) generalizes known results assume $\rho = \varphi$. This implies that $\varphi \in S(\varphi)^{1/2}$ and so φ is refinable. Assuming the normalization $\hat{\varphi}(0) = 1$ implies that $P(0) = 1$ and we recover from (6.38) a formula for the constant C_φ^-

$$(C_\varphi^-)^2 = \frac{|P^{(m)}(\pi)|^2 [\hat{\varphi}, \hat{\varphi}](\pi)}{(m!)^2 (2^{2m} - 1)}. \tag{6.39}$$

Formula (6.39) for the refinable case is exactly the formula reported in [BU3] (equation (26) therein).

6.5 Approximation properties of the non-stationary Cascade wavelets

The first simple results of this section verify that the application of a cascade operator with good properties on a given function with good approximation properties gives a function that inherits these properties. In some sense these results are connected to the known so called "zero conditions on the mask symbol" (see section 3.2 in [JP]). As we shall see, the main difference with previous work is that we use "zero conditions" on the cascade mask when operating on non-refinable functions.

Lemma 6.29 Assume $\rho_0 \in E_m(\mathbb{R}^d)$ satisfies the SF conditions of order m and let $P \in \Pi_N$ be a trigonometric polynomial defined by

$$P(w) = R(w)\prod_{r=1}^{d}\left(\frac{1+e^{-iw_r}}{2}\right)^{m_r} = \frac{1}{2}\sum_{|k|\le N} p_k e^{-ikw}, \tag{6.40}$$

with $m_r \ge m$, $r = 1,\ldots,d$ and $R(0) \ne 0$. Then ρ_1 defined by

$$\rho_1 = \sum_{|k|\le N} p_k \rho_0(2\cdot - k), \tag{6.41}$$

is in $E_m(\mathbb{R}^d)$ and satisfies the SF conditions of order m.

Proof Since $P \in \Pi_N$, the sum in (6.41) is finite so that $\rho_1 \in E_m(\mathbb{R}^d)$ and $\hat{\rho}_0, \hat{\rho}_1 \in C^m(\mathbb{R}^d)$. We now show that ρ_1 satisfies the SF conditions (6.3). Since ρ_0 satisfies SF, we have that $\hat{\rho}_1(0) = R(0)\hat{\rho}_0(0) \ne 0$. Let $l \in \mathbb{N}^d$, with $0 \le |l| \le m-1$. We use the two-scale relation and the multivariate form of Leibniz' rule to compute the partial derivatives of $\hat{\rho}_1$ at the points $2\pi n$, $n \in \mathbb{Z}^d$

$$\begin{aligned} D^l \hat{\rho}_1\Big|_{w=2\pi n} &= \frac{\partial x}{\partial x_1^{l_1}\cdots\partial x_d^{l_d}} P\left(\frac{w}{2}\right)\hat{\rho}_0\left(\frac{w}{2}\right)\Bigg|_{w=2\pi n} \\ &= 2^{-|l|}\sum_{\substack{s=(s_1,\ldots,s_d)\\ s\ne 0, s_r \le l_r}}\left(\prod_{r=1}^{d}\binom{l_r}{s_r}\right)\frac{\partial x}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}P\Bigg|_{w=\pi n}\frac{\partial x}{\partial x_1^{l_1-s_1}\cdots\partial x_d^{l_d-s_d}}\hat{\rho}_0\Bigg|_{w=\pi n}. \end{aligned}$$

Assume $n = 2n'$ for some $n' \in \mathbb{Z}^d$. As ρ_0 satisfies SF we have that

$$\frac{\partial x}{\partial x_1^{l_1-s_1}\cdots\partial x_d^{l_d-s_d}}\hat{\rho}_0\Bigg|_{w=\pi n} = \frac{\partial x}{\partial x_1^{l_1-s_1}\cdots\partial x_d^{l_d-s_d}}\hat{\rho}_0\Bigg|_{w=2\pi n'} = 0.$$

Else there exists $1 \le \gamma \le d$ such that $n = (n_1, \ldots, n_d)$ with $n_\gamma = 2k+1$, $k \in \mathbb{Z}$. Since for any $q = (q_1, \ldots, q_d) \in \mathbb{Z}^d$ with $0 \le q_r \le m_r - 1$

$$\prod_{r=1}^{d} \left(\left(\frac{1+e^{-iw_r}}{2} \right)^{m_r} \right)^{(q_r)} \Bigg|_{w_r = \pi n_r} = \left(\left(\frac{1+e^{-iw_\gamma}}{2} \right)^{m_\gamma} \right)^{(q_\gamma)} \Bigg|_{w_\gamma = \pi(2k+1)} \cdot \prod_{r \ne \gamma} \left(\left(\frac{1+e^{-iw_r}}{2} \right)^{m_r} \right)^{(q_r)} \Bigg|_{w_r = \pi n_r} = 0,$$

we obtain

$$\frac{\partial x}{\partial x_1^{s_1} \cdots \partial x_1^{s_1}} P \Bigg|_{w=\pi n} = \sum_{\substack{q=(q_1,\ldots,q_d) \\ q \ne 0, q_r \le s_r}} \left(\prod_{r=1}^{d} \binom{s_r}{q_r} \right) \left(\prod_{r=1}^{d} \left(\left(\frac{1+e^{-iw_r}}{2} \right)^{m_r} \right)^{(q_r)} \Bigg|_{w_r = \pi n_r} \right) \frac{\partial x}{\partial x_1^{s_1 - q_1} \cdots \partial x_d^{s_d - q_d}} R \Bigg|_{w=\pi n} = 0.$$

Thus, $D^l \hat{\rho}_1 \big|_{w=2\pi n} = 0$ for all $|l| \le m-1$, $n \ne 0$ and ρ_1 satisfies the Strang-Fix conditions.

♦

It is known ([R1]) that any compactly supported univariate generator $\phi \in L_2(\mathbb{R})$ that provides approximation order m is of type $\phi = N_m * f$, where N_m is the B-spline of order m and f some compactly supported tempered distribution. Thus, the smallest support possible for a given approximation order m is m. Next we see that the B-spline cascade operator can help to preserve this optimal feature.

Corollary 6.30 Assume that $\rho_0 \in L_\infty(\mathbb{R})$ satisfies the SF and summation conditions of order m and has (minimal) support size m. Then there exists $\rho_1 \in S(\rho_0)^{1/2}$ that provides approximation order m and has (minimal) support size m.

Proof Observe that by Theorem 6.10 ρ_0 provides approximation order m. We can assume that $supp(\phi) \subseteq [0, m]$ (we can always shift the construction below to this interval and then back). Select P_{N_m}, the (minimally supported) two-scale symbol of the B-Spline of order m, defined by

$$P_{N_m}(w) = \left(\frac{1+e^{-iw}}{2} \right)^m = \frac{1}{2} \sum_{k=0}^{m} p_k e^{-ikw}, \quad p_k = 2^{-m+1} \binom{m}{k}. \tag{6.42}$$

Clearly, for P_{N_m} condition (6.40) holds. Thus by Lemma 6.29, ρ_1 defined by

$$\hat{\rho}_1(w) = P_{N_m}\left(\frac{w}{2} \right) \hat{\rho}_0\left(\frac{w}{2} \right),$$

is in $L_\infty(\mathbb{R})$, has compact support and satisfies the SF and summation conditions of order m and so provides approximation order m. Also, since $p_k = 0$ for all $k \neq 0,\ldots,m$, ρ_1 has the required (minimal) support property.

♦

Thus, we see that a good cascade operator is actually an algorithm to extract a superfunction ρ_1 from the FSI space $S(\rho_0)^{1/2}$. We need to verify that the cascade process preserves approximation properties in a uniform sense. The next result overcomes this technical point using the general tools developed at the beginning of the chapter.

Corollary 6.31 Let ρ_0 be a univariate function with compact support that satisfies the SF and summation conditions of order m. Let P be a finite mask of type (6.40) associated with a cascade operator $\mathbb{C}$ and a refinable function $\phi \in L_\infty(\mathbb{R})$ and let $\rho_j := \mathbb{C}^j \rho_0$ such that,

1. $\|\rho_j\|_\infty \le M$ for all $0 \le j < \infty$,
2. $supp(\rho_j) \subseteq [-L, L]$ for all $0 \le j < \infty$.

Then the following hold,

1. There exists a constant $\tilde{C}_1$ such that for any $f \in W_p^m(\mathbb{R})$, $1 \le p \le \infty$

$$E\left(f, S(\rho_j)^h\right)_p \le \tilde{C}_1 h^m \left\|f^{(m)}\right\|_{L_p(\mathbb{R})}.$$

2. There exists a constant $\tilde{C}_2$ such that for any $f \in L_p(\mathbb{R})$, $1 \le p \le \infty$

$$E\left(f, S(\rho_l)^h\right)_p \le \tilde{C}_2 \omega_m(f,h)_p.$$

Proof It is easy to see that under our assumptions, conditions (1) and (2) of Theorem 6.10 hold. Also using Lemma 6.29 we have that for all functions in the sequence the SF conditions of order m hold and so condition (3) of Theorem 6.10 is also fulfilled. We now apply Theorem 6.10 to obtain the required estimates.

♦

Remark It is interesting to observe that for the last result we did not require that the cascade process converges, but only that it remains bounded in some box in $\mathbb{R}^2$.

For a finer analysis of the inheritance of approximation properties through the cascade process we wish to inspect the sharp constants of type (6.26).

Lemma 6.32 Let $m \geq 0$ and assume that $\rho_j \underset{L_2(\mathbb{R}^d)}{\rightarrow} \phi$ where $\phi, \rho_j \in L_2(\mathbb{R}^d)$ such that $supp(\phi), supp(\rho_j) \subseteq \Omega$ where Ω is some bounded domain. Then for any $m \geq 0$ we have the convergence of the sharp approximation constants

$$C_{\rho_j}^{-} = \frac{1}{m!}\sum_{k\neq 0}\left|\hat{\rho}_j^{(m)}(2\pi k)\right|^2 \rightarrow \frac{1}{m!}\sum_{k\neq 0}\left|\hat{\phi}^{(m)}(2\pi k)\right|^2 = C_{\phi}^{-}. \tag{6.43}$$

Proof By Lemma 2.13 we have the following convergence for any $w \in \mathbb{T}^d$

$$\sum_{k\in\mathbb{Z}^d}\left|\hat{\rho}_j^{(m)}(w+2\pi k)\right|^2 \rightarrow \sum_{k\in\mathbb{Z}^d}\left|\hat{\phi}^{(m)}(w+2\pi k)\right|^2. \tag{6.44}$$

In particular we have (6.44) for $w = 0$. It is easy to verify that $\hat{\rho}_j^{(m)}(0) \rightarrow \hat{\phi}^{(m)}(0)$. Thus, we obtain convergence also for the restricted sums (6.43).

♦

Lemma 6.33 Let $\rho_j \underset{L_2(\mathbb{R}^d)}{\rightarrow} \phi$ such that ϕ is stable and $\mathrm{supp}(\phi), \mathrm{supp}(\rho_j) \subseteq \Omega$ where Ω is some bounded domain in $\mathbb{R}^d$. Then $\left\|\Lambda_{\rho_j} - \Lambda_{\phi}\right\|_{L_\infty(\mathbb{R}^d)} \rightarrow 0$, where for any $f \in L_2(\mathbb{R}^d)$, Λ_f is the error kernel (6.25).

Proof From the continuity of the Fourier transform, is easy to see that $\left\|\hat{\rho}_j - \hat{\phi}\right\|_{C(\mathbb{R}^d)} \rightarrow 0$. By Lemma 2.13 we also have uniform convergence of the auto-correlations $\left\|\left[\hat{\rho}_j, \hat{\rho}_j\right] - \left[\hat{\phi}, \hat{\phi}\right]\right\|_{C(\mathbb{T}^d)} \rightarrow 0$. In particular, since ϕ is stable, $\left[\hat{\rho}_j, \hat{\rho}_j\right]$ are uniformly bounded from below for some $j \geq J_0$. This implies the uniform convergence of the error kernels.

♦

Corollary 6.34 Let $\phi, \rho_0 \in L_2(\mathbb{R}^d)$ be compactly supported such that ϕ is also stable. Then if ϕ, ρ_0 provide approximation order m then there exists a constant C^{-} such that for any $f \in H^r(\mathbb{R}^d)$

$$E\left(f, S(\rho_j)^h\right)_2 \leq C^{-} h^m |f|_{H^m(\mathbb{R}^d)} + O(h^r), \qquad j \geq 0.$$

Proof This is immediate consequence of Lemma 6.32, Lemma 6.33 and Theorem 6.17.

♦

An important application of the discussion so far is the following result.

Theorem 6.35 Let $\{\rho_j\}_{j\ge 0}$ be defined by

$$\hat{\rho}_{j+1}(w) = P_{N_m}\left(\frac{w}{2}\right)\hat{\rho}_j\left(\frac{w}{2}\right), \quad j \ge 1.$$

Where

1. P_{N_m} is the B-spline two-scale symbol (6.42),
2. ρ_0 provides approximation order m,
3. ρ_0 has (minimal) support size m.
4. $\left|\left(\hat{\rho}_0 - \hat{N}_m\right)(w)\right| = O(|w|)$ near the origin.

Then

1. For any $1 \le p \le \infty$, the sequence ρ_j converges to the B-spline N_m in any p norm,
2. Each ρ_j has (minimal) support size m,
3. There exists a constant $\tilde{C}$ such that for any $f \in W_p^m(\mathbb{R})$, $1 \le p \le \infty$

$$E\left(f, S(\rho_j)^h\right)_p \le \tilde{C}h^m |f|_{W_p^m(\mathbb{R})}, \qquad j \ge 0.$$

4. The sharp constants $C_{\rho_j}^-$ converge to $C_{N_m}^-$. Also there exist a constant C^- such that for any function $f \in H^r(\mathbb{R})$

$$E\left(f, V_j^h\right)_2 \le C^- h^m |f|_{H^m(\mathbb{R})} + O(h^r), \qquad j \ge 0.$$

Proof

1. We use the cascade result Theorem 5.12.
2. We use Corollary 6.30.
3. We use Theorem 6.10.
4. We use Corollary 6.34.

♦

Example 6.36 Let $\rho_0 := OM_4$ and $\phi = N_4$ (see Example 6.6). Let $\{\rho_j\}$ be the sequence constructed by Theorem 6.35. Then

$$\left(\frac{C_{N_4}^-}{C_{OM_4}^-}\right)^{\frac{1}{4}} \approx 1.45 \Rightarrow \left(\frac{C_{N_4}^-}{C_{\rho_1}^-}\right)^{\frac{1}{4}} \approx 1.07.$$

This means that the first generator constructed by the cascade process is not as good as the initial optimal $\rho_0 := OM_4$, but still much better than the B-spline. Obviously $\{\rho_j\}$ quickly converge to the B-spline and so

$$\left(\frac{C^-_{N_4}}{C^-_{\rho_j}}\right)^{\frac{1}{4}} \underset{j\to\infty}{\to} 1 .$$

The corresponding minimally supported semi-orthonormal wavelets $\{\psi_j\}$ can be calculated using (5.25). Recall that these wavelets provide for any $J \geq 0$ the stable decomposition

$$S(\rho_J) = \overset{\infty}{\underset{j=1}{\oplus}} S\left(2^{j/2}\psi_{J+j}\right)^{2^j} ,$$

which can be dilated to any scale. Therefore any approximation obtained from dilations of the PSI space $S(\rho_J)$ has a representation in the form of a non-stationary wavelet sum.

In applications such as signal processing, one usually approximates a function and then decomposes the approximation to a sum of a coarse approximation and a few wavelet subspaces. Thus, at least in theory, the non-stationary wavelets derived from a B-spline cascade multiresolution initialized by OM_4, outperform spline-wavelets [Ch], [Da] on the first levels.

♦

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