2.3. Markov-type properties

The classic Markov property of the heat kernel, also known as a **Conservative Markov Transition** and **Stochastic Completeness**, ensures that, while heat is transported by a solution to the heat equation, there is mass conservation

$$\int_M p_t(x,y)d\mu(y) \equiv 1, \quad \forall t > 0. \tag{2.42}$$

We have seen in §1.1, several examples for heat kernels satisfying the Markov property, including for example, the classical example of $M = \mathbb{R}^d$ and $L = -\Delta$, where the Markov property is satisfied by the normalized Gaussians. By analytic continuation, the Markov property implies

$$\int_M p_z(x,y)d\mu(y) \equiv 1, \quad \forall z = t + iu, t > 0. \tag{2.43}$$

LEMMA 2.22. *Assume the Markov property of the heat kernel (2.42) is satisfied. Let $\varphi \in C_0^k(\mathbb{R})$, $k > 2d$, be even. Then,*

$$\int_M \varphi(\delta\sqrt{L})(x,y)d\mu(y) = \varphi(0), \quad \forall \delta > 0, x \in M. \tag{2.46}$$

PROOF. Define $\theta(u) := \varphi(\sqrt{|u|})e^u$, $u \in \mathbb{R}$, so that $\varphi(u) = \theta(u^2)e^{-u^2}$, $u \in \mathbb{R}_+$. Under our assumptions, θ is continuous, piecewise C^k, compactly supported and $\hat{\theta} \in L^1(\mathbb{R})$. Therefore the inverse Fourier transform holds pointwise

$$\theta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\theta}(\omega)e^{i\omega x}d\omega, \quad \forall x \in \mathbb{R}.$$

This allows to proceed with

$$\begin{aligned}\varphi(\delta\sqrt{L}) &= \theta(\delta^2 L)e^{-\delta^2 L}\\ &= \int_0^\infty \theta(\delta^2\lambda)e^{-\delta^2\lambda}dE_\lambda\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^\infty \hat{\theta}(\omega)e^{-\delta^2\lambda(1-i\omega)}dE_\lambda d\omega.\end{aligned}$$

Applying further the analytical version of the Markov property of the heat kernel (2.43), gives for any $x \in M$ and $\delta > 0$

$$\begin{aligned}\int_M \varphi(\delta\sqrt{L})(x,y)d\mu(y) &= \int_M \theta(\delta^2 L)e^{-\delta^2 L}(x,y)d\mu(y)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\theta}(\omega)\left(\int_M p_{\delta^2(1-i\omega)}(x,y)d\mu(y)\right)d\omega\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\theta}(\omega)d\omega\\ &= \theta(0) = \varphi(0).\end{aligned}$$

□

THEOREM 2.23. *Assume the Markov property (2.42). Let $\theta \in C_0^\infty(\mathbb{R})$ be even, with* $\operatorname{supp}(\theta) \subset [-R, R]$, *for some $R > 0$, and $\theta(0) = 1$. Then, for any $f \in L^p(M)$, $1 \le p < \infty$ and for any $f \in UCB$, one has*

$$f \underset{L^p}{=} \lim_{\delta \to 0} \theta(\delta\sqrt{L}) f. \tag{2.47}$$

PROOF. By Theorem 2.16, $\theta(\delta\sqrt{L})$ is an integral operator with a kernel satisfying for $k > 2d$

$$|\theta(\delta\sqrt{L})(x,y)| \le c D_{\delta,k}(x,y) \le c|B(x,\delta)|^{-1} \left(1 + \frac{\rho(x,y)}{\delta}\right)^{-k+d/2},$$

where for the last inequality we used (1.30). For any $x \in M$ and $r > 0$, suppose $2^{J-1}\delta \le r \le 2^J\delta$. Then, using the above along with the doubling condition (1.16) provides

$$
\begin{aligned}
&\int_{M\setminus B(x,r)} |\theta(\delta\sqrt{L})(x,y)|d\mu(y) \\
&\quad \le c|B(x,\delta)|^{-1}\sum_{j=J}^{\infty}\int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-k+d/2} d\mu(y) \\
&\quad \le c|B(x,\delta)|^{-1}\sum_{j=J}^{\infty}|B(x,2^j\delta)|(1+2^j)^{-k+d/2} \\
&\quad \le c\sum_{j=J}^{\infty}2^{jd}(1+2^j)^{-k+d/2} \\
&\quad \le c2^{-J(k-3d/2)} \le c\left(\frac{\delta}{r}\right)^{k-3d/2}.
\end{aligned}
$$

Therefore, uniformly in M, for any $r > 0$

$$\int_{M\setminus B(x,r)} |\theta(\delta\sqrt{L})(x,y)|d\mu(y) \underset{\delta\to 0}{\longrightarrow} 0\,. \tag{2.48}$$

At the same time, since $\theta(0) = 1$, by (2.46)

$$\int_M \theta(\delta\sqrt{L})(x,y)d\mu(y) = 1, \quad \forall\delta > 0, x \in M.$$

Using these properties it is easy to prove (2.47) for any uniformly continuous function using the same technique used for summability kernels in the Euclidean case. Let us show this for the sake of completeness. First, observe that by (1.33) and (2.28) there exists a constant $\tilde{c} > 0$, depending on θ, such that

$$\|\theta(\delta\sqrt{L})(x,\cdot)\|_1 \leq \tilde{c}, \quad \forall x \in M, 0 < \delta < 1. \tag{2.49}$$

Assume $f : M \to \mathbb{C}$ is uniformly continuous, $f \neq 0$. Therefore, for any $\varepsilon > 0$ there exists $r > 0$, such that for any $x, y \in M$, $\rho(x, y) < r$, we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2\tilde{c}}.$$

At the same time, for this fixed $r > 0$, by (2.48), we may select $\delta > 0$ sufficiently small, such that

$$\int_{M \setminus B(x,r)} |\theta(\delta\sqrt{L})(x, y)| d\mu(y) < \frac{\varepsilon}{4\|f\|_\infty}, \quad \forall x \in M.$$

This yields for any $x \in M$, using (2.48) and (2.49)

$$
\begin{aligned}
&|f(x) - \theta(\delta\sqrt{L})f(x)| \\
&\quad = \left| \int_M (f(x) - f(y))\theta(\delta\sqrt{L}(x,y)d\mu(y) \right| \\
&\quad \leq \int_{B(x,r)} |f(x) - f(y)||\theta(\delta\sqrt{L}(x,y)|d\mu(y) + \int_{M\setminus B(x,r)} |f(x) - f(y)||\theta(\delta\sqrt{L}(x,y)|d\mu(y) \\
&\quad \leq \frac{\varepsilon}{2\tilde{c}} \int_M |\theta(\delta\sqrt{L}(x,y)|d\mu(y) + 2\|f\|_\infty \int_{M\setminus B(x,r)} |\theta(\delta\sqrt{L}(x,y)|d\mu(y) \\
&\quad \leq \varepsilon.
\end{aligned}
$$

The proof for the case $1 \leq p < \infty$ follows from the density of uniformly continuous compactly supported functions in L^p, $1 \leq p < \infty$. □

Frames

4.1. Frames in $\mathbf{L^2}$

One of the main goals of function space theory is to construct representations that can be used for approximation in the corresponding $L^p(M)$ norms, as well as provide simple discretized characterizations of the function spaces: L^p, Hardy, Sobolev, Besov and Triebel-Lizorkin. Perhaps the main starting point for a suitable representation is to serve as a stable basis of the $L^2(M)$ space. Then, generally speaking, if the elements of the representation also have good regularization and localization properties, then approximation in other p-norms as well as characterization of the function spaces is possible.

Classical wavelets [12],[20] are bases of $L_2(\mathbb{R}^d)$ that are well-localized with respect to the Euclidean metric in space and frequency. Wavelet constructions have many applications in harmonic analysis, approximation theory, function space theory, as well as signal processing and numerical methods for PDEs. The simplest example is the univariate Haar orthonormal basis which is perfectly localized in space since it is compactly supported. It is defined through dilations and translations $\{\psi_{j,k}\}$, $\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k)$, $j,k \in \mathbb{Z}$, where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is the 'mother' wavelet

$$\psi(x) := \begin{cases} 1, & 0 \leq x < 1/2, \\ -1, & 1/2 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.1. A countable family of elements $\{\psi_i\}_{i\in I}$, contained in a Hilbert space $\mathcal{H}$, is a **frame** if there exist constants $0 < A \leq B < \infty$ such that for any $f \in \mathcal{H}$

$$A\,\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in I} |\langle f, \psi_i\rangle_{\mathcal{H}}|^2 \leq B\,\|f\|_{\mathcal{H}}^2\,. \tag{4.1}$$

The upper frame bound means that $\{\psi_i\}_{i\in I}$ is a **Bessel Sequence** and that the operator $T : l^2(I) \to \mathcal{H}$ defined by

$$T(\{c_i\}) := \sum_{i\in I} c_i\psi_i,$$

is bounded. Its adjoint $T^* : \mathcal{H} \to l^2(I)$ is

$$T^*f = \{\langle f, \psi_i\rangle\}_{i\in I}.$$

Composing T with its adjoint T^* give the **Frame Operator** $S : \mathcal{H} \to \mathcal{H}$

$$Sf = TT^*f = \sum_{i\in I}\langle f, \psi_i\rangle\psi_i, \quad \forall f \in \mathcal{H}.$$

One can show that the frame operator is bounded and surjective, which leads to the frame decomposition

$$f = SS^{-1}f = \sum_{i \in I} \langle f, S^{-1}\psi_i \rangle \psi_i, \quad \forall f \in \mathcal{H}.$$

One then may choose $\{\tilde{\psi}_i := S^{-1}\psi_i\}_{i \in I}$ as the **Canonical Dual Frame** for $\{\psi_i\}_{i \in I}$ that provides the representation

$$f = \sum_{i \in I} \langle f, \tilde{\psi}_i \rangle \psi_i, \quad \forall f \in \mathcal{H}.$$

4.2. Construction of Well Localized Frames

For the following construction, we assume the reverse doubling condition (1.18) and the Markov property (2.42) hold.

4.2.1. Construction of a Littlewood-Paley Type Frame. Littlewood and Paley initiated a fundamental branch of harmonic analysis in the 30s, where the Fourier series is split into dyadic blocks $f = \sum_j \Delta_j(f)$, and then most functional spaces can be characterized by size estimates on $\Delta_j(f)$. G. David, J. L. Journe and S. Semmes [**13**] used an idea of R. Coifman to generalize the Littlewood-Paley analysis to the general setting on spaces of homogeneous type. One constructs [**19**] useful wavelet representations of functions based on the approximation to the identity.

To this end, we construct two admissible cut-off functions as per Definition 2.27. First, we select $b > 1$, sufficiently large, such that $b^{3/4} \geq \beta$, where β is from Definition 2.27. Now let $\Phi \in C^\infty(\mathbb{R})$, be even, with the following properties:

(i) $\Phi \equiv 1$ on $[-1, 1]$,
(ii) $0 \leq \Phi \leq 1$,
(iii) $\mathrm{supp}(\Phi) \subset [-b, b]$,

Set $\Psi := \Phi - \Phi(b\cdot)$. We note the following properties of Ψ:

(i) $0 \leq \Psi \leq 1$,
(ii) $\mathrm{supp}(\Psi) \subset [-b, -b^{-1}] \cup [b^{-1}, b]$,
(iii) We also assume Φ is selected such that there exist $\tilde{c} > 0$, so that $\Psi(u) \geq \tilde{c}$, for $u \in [-b^{3/4}, b^{-3/4}] \cup [b^{-3/4}, b^{3/4}]$.

Set

$$\Psi_0 := \Phi, \qquad \Psi_j := \Psi(b^{-j}\cdot), \quad j \geq 1. \tag{4.2}$$

Clearly, $\Psi_j \in C^\infty(\mathbb{R})$, $0 \leq \Psi_j \leq 1$, $\mathrm{supp}(\Psi_0) \subset [-b, b]$, $\mathrm{supp}(\Psi_j) \subset [-b^{j+1}, -b^{j-1}] \cup [b^{j-1}, b^{j+1}]$, $j \geq 1$, and

$$\sum_{j\geq 0} \Psi_j = \Phi + \Phi(b^{-1}\cdot) - \Phi + \Phi(b^{-2}\cdot) - \Phi(b^{-1}\cdot) + \cdots \equiv 1.$$

We would like to show that $\{\Psi_j\}_{j\geq 0}$ provide a Littlewood-Paley decomposition. That is, for any $f \in L^p(M)$, $1 \leq p < \infty$ and for a uniformly continuous bounded function $f \in UCB$, equipped with the L^∞ norm

$$f \underset{L^p}{=} \sum_{j=0}^{\infty} \Psi_j(\sqrt{L}) f. \tag{4.3}$$

THEOREM 4.3. *The set* $\{\Psi_j\}_{j\geq 0}$ *defined by* (4.2), *provides the Littlewood-Paley decomposition* (4.3).

PROOF. Let $\theta := \Psi_0 + \Psi_1$. Then, by the construction $\theta = \Phi + \Phi(b^{-1}\cdot) - \Phi = \Phi(b^{-1}\cdot)$, We see that θ satisfies the conditions of Theorem 2.23 and so the operators $\{\theta(\delta\sqrt{L})\}_{\delta>0}$ provide an approximation of identity in $L^p(M)$. Next, observe that for any $J \geq 0$, $\sum_{j=0}^{J+1} \Psi_j = \theta(b^{-J}\cdot)$. Therefore, for any $f \in L^p$

$$\lim_{J\to\infty} \sum_{j=0}^{J+1} \Psi_j(\sqrt{L})f = \lim_{J\to\infty} \theta(b^{-J}\sqrt{L})f = f.$$

□

Theorem 4.4. *The set* $\{\Psi_j\}_{j\geq 0}$ *defined by* (4.2) *satisfies*

$$\frac{1}{2}\|f\|_2^2 \leq \sum_{j=0}^{\infty} \|\Psi_j(\sqrt{L})f\|_2^2 \leq \|f\|_2^2, \quad \forall f \in L^2(M). \tag{4.4}$$

Proof. Let $u \geq 0$. Observe that since $0 \leq \Psi_j \leq 1$, for all $j \geq 0$, we get

$$\sum_{j=0}^{\infty} \Psi_j^2(u) \leq \sum_{j=0}^{\infty} \Psi_j(u) = 1, \quad \forall u \in \mathbb{R}.$$

Next, observe that from the properties of $\{\Psi_j\}_{j\geq 0}$, any $u \geq 0$, is in the support of at most two function Ψ_j and Ψ_{j+1}, for some $j \geq 0$. Thus

$$
\begin{aligned}
1 &= \sum_{j=0}^{\infty} \Psi_j(u) \sum_{j=0}^{\infty} \Psi_j(u) \\
&= \sum_{j=0}^{\infty} \Psi_j^2(u) + \sum_{j=0} \Psi_j(u)\Psi_{j+1}(u) \\
&\leq \sum_{j=0}^{\infty} \Psi_j^2(u) + \sum_{j=0} \max(\Psi_j(u), \Psi_{j+1}(u))^2 \\
&\leq 2\sum_{j=0}^{\infty} \Psi_j^2(u).
\end{aligned}
$$

This yields

$$
\frac{1}{2} \leq \sum_{j=0}^{\infty} \Psi_j^2(u) \leq 1, \quad \forall u \in \mathbb{R}. \tag{4.5}
$$

Next observe that for any $f \in L^2$

$$\begin{aligned}\sum_{j=0}^{\infty} \|\Psi_j(\sqrt{L})f\|_2^2 &= \sum_{j=0}^{\infty}\langle \Psi_j^2(\sqrt{L})f, f\rangle \\ &= \int_0^{\infty} \left(\sum_{j=0}^{\infty} \Psi_j^2(\sqrt{\lambda})\right) d\langle E_\lambda f, f\rangle \\ &= \int_0^{\infty} \left(\sum_{j=0}^{\infty} \Psi_j^2(\sqrt{\lambda})\right) d\|E_\lambda f\|_2^2.\end{aligned}$$

Applying (1.47) and (4.5) gives (4.4). □