Introduction to function spaces: Assignment II

1. Prove that for $\eta, \varphi \in S$, $t, s > 0$

$$\left(\eta * \varphi_s\right)_t (z) = \eta_t * \varphi_{st}(z), \qquad z \in \mathbb{R}^n.$$

2. A cube in $\mathbb{R}^n$,

$$Q = 2^{-j}\left([0,1]^n + k\right) = \left[2^{-j}k_1, 2^{-j}(k_1+1)\right] \times \cdots \times \left[2^{-j}k_n, 2^{-j}(k_n+1)\right], \qquad j \in \mathbb{Z},$$

is called a dyadic cube (at scale j). Define the Dyadic Hardy-Littlewood maximal function by

$$M_d f(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x)|.$$

Prove that M_d maps $L_p \to L_p$, $1 < p \le \infty$ and $L_1 \to L_{1,\infty}$.

Hint No need to go through the proof of the maximal theorem.

3. Let $T : L_1(X) \to L_{1,\infty}(X)$, with norm $\|T\|_{L_1 \to L_{1,\infty}} < \infty$ and suppose that $f \in L_1(X)$. Prove that for any set $A \subset X$ of finite measure (volume) and $0 < q < 1$

$$\int_A |Tf(x)|^q \le \frac{1}{1-q} \|T\|^q |A|^{1-q} \|f\|_1^q.$$

Hint: Recall that $\int_A |Tf(x)|^q = q\int_0^\infty \alpha^{q-1} \left|\left\{x \in A : |Tf(x)| > \alpha\right\}\right| d\alpha$. Split the right side integral into $\int_0^\infty = \int_0^\lambda + \int_\lambda^\infty$ with $\lambda = |A|^{-1} \|T\| \|f\|_1$.

4. Can you prove the Vitali covering lemma for triangles in $\mathbb{R}^2$? That is, does there exists an absolute constant $c > 0$, such that for any union of triangles $E = \bigcup_{j=1}^N \Delta_j$, there exist a pairwise interior disjoint sub-collection $\left\{\Delta_{j_k}\right\}_{k=1}^M$, such that $\left|\bigcup_{k=1}^M \Delta_{j_k}\right| = \sum_{k=1}^M \left|\Delta_{j_k}\right| \ge c|E|$.