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On Dual Spaces Of Anisotropic Hardy Spaces

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In this paper we generalize the analysis of the Campanato-type dual spaces [1], [3], for the highly anisotropic Hardy spaces on $\mathbb{R}^n$ introduced in [8]. These Hardy spaces are constructed over multi-level ellipsoid covers of $\mathbb{R}^n$, where the ellipsoids can change shape rapidly from point to point and from level to level.

1 Introduction

The theory of real Hardy spaces in more 'geometric' settings has also received much attention. Coifman and Weiss pioneered this field in the 70s [4], [5]. Then, Folland and Stein in the 80s studied Hardy spaces over homogeneous groups [12]. However, in general settings, such as the setting of spaces of homogeneous type, the Hardy Spaces with p 'close' to zero do not have sufficient structure. Bownik [1] investigated a special form of Hardy spaces defined over $\mathbb{R}^n$, where the Euclidian balls are replaced by images of the unit ball by powers of a fixed expansion matrix. In this setup, Bownik was able to construct and fully analyze anisotropic Hardy spaces for the full range $0 < p \leq 1$. In [8], these spaces were significantly generalized by constructing Hardy spaces $H^p(\Theta)$, $0 < p \leq 1$, over ellipsoid multi-level covers Θ, where the anisotropy may change rapidly from point to point.

Here, motivated by the careful analysis of [1], [3], we investigate the Campanato-type dual spaces of $H^p(\Theta)$, $0 < p < 1$ (for $p = 1$, it follows from the general theory of spaces of homogeneous type that the dual space is the suitable BMO space [8]). In Section 2 we recall the basic properties of the anisotropic multi-level ellipsoid covers introduced in [6]. The ellipsoid covers induce anisotropic quasi-distances on $\mathbb{R}^n$ and together with the usual Lebesgue measure, form spaces of homogeneous type. In Section 3 we review multiresolution anisotropic representations [7] that are required for the proof of our main result. In Section 4 we review the anisotropic Hardy spaces $H^p(\Theta)$ [8]. As in the classical case, $H^p(\Theta) \sim L^p(\mathbb{R}^n)$ for any $1 < p \leq \infty$, and any continuous ellipsoid cover Θ and therefore we focus our attention on the case $0 < p \leq 1$. In Section 5 we define anisotropic Campanato-type spaces and prove that, modulo a space of polynomials of certain degree, they are indeed the duals of $H^p(\Theta)$, $0 < p < 1$. The main technical issue one needs to deal with is that ,in general, a linear functional that is uniformly bounded on atoms is not automatically bounded on the Hardy space [2].

Throughout the paper, the constants $c > 0$, depend on various fixed constants such as the parameters of our covers, the dimension n as well as other parameters and their value may change from line to line.

2 Anisotropic ellipsoid covers of $\mathbb{R}^n$

We recall the definitions of [6]. An ellipsoid is the image of the Euclidian unit ball B^* in $\mathbb{R}^n$ via an affine transform. For a given ellipsoid θ, we let A_θ be an affine transform such that $\theta = A_\theta(B^*)$. Denoting by $v_\theta := A_\theta(0)$ the center of θ we have

$$A_\theta(x) = M_\theta x + v_\theta,$$

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where M_θ is a nonsingular $n \times n$ matrix.

Definition 2.1 We say that

$$\Theta := \bigcup_{t \in \mathbb{R}} \Theta_t$$

is a **continuous ellipsoid cover** of $\mathbb{R}^n$ if it satisfies the following conditions, where $p(\Theta) := \{a_1, \ldots, a_6\}$ are positive constants:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ there exists an ellipsoid $\theta(x,t) \in \Theta_t$ and an affine transform $A_{x,t}(y) = M_{x,t}(y) + x$ such that $\theta(x,t) = A_{x,t}(B^*)$ and

$$a_1 2^{-t} \leq |\theta(x,t)| \leq a_2 2^{-t}. \tag{2.1}$$

(ii) For any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \geq 0$, if $\theta(x,t) \cap \theta(y, t+s) \neq \emptyset$, then

$$a_3 2^{-a_4 s} \leq 1 \big/ \left\| M_{y,t+s}^{-1} M_{x,t} \right\| \leq \left\| M_{x,t}^{-1} M_{y,t+s} \right\| \leq a_5 2^{-a_6 s}. \tag{2.2}$$

Definition 2.2 We call

$$\Theta = \bigcup_{m \in \mathbb{Z}} \Theta_m,$$

a **discrete multilevel ellipsoid cover** of $\mathbb{R}^n$ if the following conditions are obeyed, where $p(\Theta) := \{a_1, \ldots, a_8\}$ are positive constants:

(a) Every level Θ_m, $m \in \mathbb{Z}$, consists of ellipsoids θ such that

$$a_1 2^{-m} \leq |\theta| \leq a_2 2^{-m}, \tag{2.3}$$

and Θ_m is a cover of $\mathbb{R}^n$, i.e. $\mathbb{R}^n = \bigcup_{\theta \in \Theta_m} \theta$.

(b) For each $\theta \in \Theta$ let A_θ be an affine transform associated with θ, of the form

$$A_\theta(y) = M_\theta y + v_\theta, \quad M_\theta \in \mathbb{R}^{n \times n},$$

such that $\theta = A_\theta(B^*)$ and $v_\theta = A(0)$ is the center of θ. We postulate that for any $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+\nu}$, $\nu \geq 0$, with $\theta \cap \theta' \neq \emptyset$, we have

$$a_3 2^{-a_4 \nu} \leq 1 \Big/ \left\| M_{\theta'}^{-1} M_\theta \right\|_{l_2 \to l_2} \leq \left\| M_\theta^{-1} M_{\theta'} \right\|_{l_2 \to l_2} \leq a_5 2^{-a_6 \nu} \tag{2.4}$$

(c) Each $\theta \in \Theta_m$ can intersect with at most N_1 ellipsoids from Θ_m.

(d) For any $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}$ there exists $\theta \in \Theta_m$ such that $x \in \theta^\diamondsuit$, where $\theta^\diamondsuit$ is the dilated version of θ by a factor of $a_7 < 1$, i.e. $\theta^\diamondsuit = A_\theta(B(0, a_7))$.

(e) If $\theta \cap \eta \neq \emptyset$ with $\theta \in \Theta_m$ and $\eta \in \Theta_m \cup \Theta_{m+1}$, then $\theta^\diamondsuit \cap \eta^\diamondsuit \neq \emptyset$, where $\theta^\diamondsuit, \eta^\diamondsuit$ are the dilated versions of θ, η by a factor a_7 as above.

We refer the reader to [6] and [8] for discussions and examples of highly anisotropic covers. We shall need the fact that a continuous cover can be 'sampled' to create a discrete cover with equivalent ellipsoids

Theorem 2.3 *[6] For every continuous ellipsoid cover Θ of $\mathbb{R}^n$ there is a discrete ellipsoid cover $\widehat{\Theta}$ such that each ellipsoid $\theta \in \widehat{\Theta}$ is obtained from an ellipsoid in Θ by dilation by a factor r_θ obeying $(a_7+1)/2 \leq r_\theta \leq 1$.*

The ellipsoid covers induce quasi-distances on $\mathbb{R}^n$. A *quasi-distance* on a set X is a mapping $\rho : X \times X \to [0, \infty)$ that satisfies the following conditions for all $x, y, z \in X$:

(a) $\rho(x, y) = 0 \Leftrightarrow x = y$,

(b) $\rho(x, y) = \rho(y, x)$,

(c) For some $\kappa \geq 1$

$$\rho(x, y) \leq \kappa(\rho(x, z) + \rho(z, y)).$$

Let Θ be a cover. We define $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$\rho(x, y) = \inf_{\theta \in \Theta} \{|\theta| : x, y \in \theta\}. \tag{2.5}$$

Theorem 2.4 *[6] The function ρ in (2.5), induced by an ellipsoid cover, is a quasi-distance on $\mathbb{R}^n$.*

Let Θ be an ellipsoid cover inducing a quasi-distance ρ. We denote

$$B(x, r) := \{y \in \mathbb{R}^n : \rho(x, y) < r\}.$$

Evidently,

$$B(x, r) = \bigcup_{\theta \in \Theta} \{\theta : |\theta| < r, \ x \in \theta\}.$$

Theorem 2.5 *[6] Let Θ be an ellipsoid cover. For each ball $B(x, r)$, there exist ellipsoids $\theta', \theta'' \in \Theta$, such that $\theta' \subset B(x, r) \subset \theta''$ and $|\theta'| \sim |B(x, r)| \sim |\theta''| \sim r$, where the constants depend on $p(\Theta)$.*

Spaces of homogeneous type were first introduced in [4] (see [11]) as a means to extend the Calderón-Zygmund theory of singular integral operators to more general settings. Let X be a topological space endowed with a Borel measure μ and a quasi-distance ρ. Assume that the balls $B(x, r) := \{y \in X : \rho(x, y) < r\}$, $x \in X$, $r > 0$, form a basis for the topology in X. The space (X, ρ, μ) is said to be of *homogenous type* if there exists a constant λ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq \lambda \mu(B(x, r)). \tag{2.6}$$

If (2.6) holds then μ is said to be a *doubling measure* [14]. A space of homogeneous type is said to be *normal*, if the equivalence $\mu(B(x, r)) \sim r$ holds. Theorem 2.5 ensures (2.6) holds for the case of an ellipsoid cover and implies that it induces a normal space of homogeneous type $(\mathbb{R}^n, \rho, dx)$, where ρ is the quasi-distance (2.5) and dx is the Lebesgue measure.

3 Anisotropic multiresolution representations

For the proof of our main result we shall need the constructions from [6] and [7]. Let Θ be a discrete ellipsoid cover (see Definition 2.2), possibly sampled from and equivalent to a continuous cover (see Theorem 2.3). We shall first construct for each level $m \in \mathbb{Z}$ a stable basis Φ_m whose elements are C^∞ 'bumps' that reproduce polynomials and are supported on the ellipsoids of Θ_m. To this end, we split Θ_m into no more than N_1 disjoint sets $\{\Theta_m^\nu\}_{\nu=1}^{N_1}$ (N_1 appears in condition (c) in Definition 2.2), so that neither two ellipsoids $\theta', \theta'' \in \Theta_m$, with $\theta' \cap \theta'' \neq \emptyset$ are of the same color. Later, where we require the stability of the 'two-level splits' of [6] (see below), we shall need a stronger coloring scheme, where two intersecting ellipsoids from adjacent levels also have different colors.

For a fixed order $r \in \mathbb{N}$, there exist functions $\phi_\nu \in C^\infty(\mathbb{R}^n)$, $1 \leq \nu \leq N_1$, with the following properties:

(i) $\phi_\nu \geq 0$ with $\operatorname{supp} \phi_\nu = \overline{B^*}$, where B^* is the Euclidean unit ball in $\mathbb{R}^n$.

(ii) The restriction of ϕ_ν is a polynomial of degree $2\nu r$ on $B(0, (a_7 + 1)/2)$.

(iii) In addition

$$\phi_\nu|_{B(0, a_7)} \geq c_1 > 0, \quad c_1 = c_1(N_1, r). \tag{3.1}$$

For any ellipsoid θ let A_θ be the affine transform satisfying $A_\theta\left(B^*\right)=\theta$ and let $\phi_\theta := \phi_\nu \circ A_\theta^{-1}$, if $\theta \in \Theta_m^\nu$. It is standard to form a partition of unity $\left\{\tilde{\phi}_\theta\right\}_{\theta\in\Theta_m}$ by setting

$$\tilde{\phi}_\theta := \frac{\phi_\theta}{\sum\limits_{\theta'\in\Theta_m}\phi_{\theta'}}. \tag{3.2}$$

Observe that property (d) of discrete ellipsoid covers (see Definition 2.2) together with (3.1) ensure that $0<c'\leq \sum_{\theta\in\Theta_m}\phi_\theta\left(x\right)\leq c''$, for all $x\in\mathbb{R}^n$ and hence $\left\{\tilde{\phi}_\theta\right\}$ are well defined and satisfy the *partition of unity*

$$\sum\nolimits_{\theta\in\Theta_m}\tilde{\phi}_\theta = 1. \tag{3.3}$$

By property (ii) above, the 'core' part of each $\tilde{\phi}_\theta$ is a rational function, whose nominator is a polynomial of a certain degree which is different from the degrees of the nominators of its neighbors, i.e. the basis functions supported on neighbor ellipsoids. This construction gives local linear independence of neighbor basis function and eventually leads to the global stability of Φ_m.

Denote by Π_{r-1} the polynomials of total degree $r-1$. Fix $1\leq\nu\leq N_1$. Suppose $\{P_\beta : \beta\in\mathbb{N}^n, |\beta|=\beta_1+\cdots+\beta_n\leq r-1\}$ is an orthonormal basis for Π_{r-1} in the weighted norm $\|f\|_{L_2(B^*,\phi_\nu)} := \|f\phi_\nu\|_{L_2(B^*)}$. Then for any $\theta\in\Theta_m^\nu$ and $\beta\in\mathbb{N}^n$, $|\beta|<r$, we define

$$P_{\theta,\beta} := |\theta|^{-1/2} P_\beta\circ A_\theta^{-1}, \tag{3.4}$$

and set

$$\varphi_{\theta,\beta} := P_{\theta,\beta}\tilde{\phi}_\theta. \tag{3.5}$$

To simplify our notation, we denote

$$\Lambda_m := \left\{\lambda := (\theta,\beta) : \theta\in\Theta_m, |\beta|<r\right\}, \tag{3.6}$$

and if $\lambda=(\theta,\beta)$ we shall denote by θ_λ and β_λ the components of λ.

Notice that from our construction $\|\varphi_\lambda\|_2 = 1$ and in general $\|\varphi_\lambda\|_p \sim |\theta_\lambda|^{1/p-1/2}$, $0\leq p\leq\infty$. In going further we define the mth level basis by

$$\Phi_m := \left\{\varphi_\lambda : \lambda\in\Lambda_m\right\},$$

and set $\mathcal{S}_m^r := \overline{\text{span}}\left(\Phi_m\right)$.

It is easy to see that $\Pi_{r-1}\subset\mathcal{S}_m^r$, since for any polynomial $P\in\Pi_{r-1}$ and $\theta\in\Theta_m$ there exist a representation $P=\sum_{|\beta|<r}c_{\theta,\beta}P_{\theta,\beta}$ and therefore, by the partition of unity (3.3)

$$P=\sum_{\theta\in\Theta_m}P\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m,|\beta|<r}c_{\theta,\beta}P_{\theta,\beta}\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m,|\beta|<r}c_{\theta,\beta}\varphi_{\theta,\beta} = \sum_{\lambda\in\Lambda_m}c_\lambda\varphi_\lambda. \tag{3.7}$$

As we already discussed, the stability of Φ_m is critical for our further development.

Theorem 3.1 *[6] For $f\in\mathcal{S}_m^r\cap L_p$, $0<p\leq\infty$, with $f=\sum\limits_{\lambda\in\Lambda_m}c_\lambda\varphi_\lambda$, the following holds*

$$\|f\|_p \sim \left(\sum_{\lambda\in\Lambda_m}\|c_\lambda\varphi_\lambda\|_p^p\right)^{1/p} \sim 2^{m\left(\frac{1}{2}-\frac{1}{p}\right)}\left(\sum_{\lambda\in\Lambda_m}|c_\lambda|^p\right)^{1/p}, \tag{3.8}$$

with the obvious modification when $p=\infty$ and where the constants of equivalency depend only on $p\left(\Theta\right), n, r, p$ and our choice of 'bumps' $\{\phi_\nu\}_{\nu=1,\ldots,N_1}$.

Let $\Omega \subseteq \mathbb{R}^n$ be a subdomain with non empty interior. For $f \in L_p(\Omega)$, $0 < p \leq \infty$, $h \in \mathbb{R}^n$ and $r \in \mathrm{N}$ we recall the rth order difference operator

$$\Delta_h^r(f,x) := \Delta_h^r(f,x,\Omega) := \begin{cases} \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x+kh) & [x,x+rh] \subset \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

where $[x,y]$ denotes the line segment connecting any two points $x,y \in \mathbb{R}^n$. The *modulus of smoothness of order* r of a function in $L_p(\mathbb{R}^n)$ is defined by

$$\omega_r(f,t)_p := \sup_{|h| \leq t} \left\| \Delta_h^r(f,\cdot) \right\|_{L_p(\mathbb{R}^n)}, \quad t > 0, \tag{3.9}$$

where for $h \in \mathbb{R}^n$, $|h|$ denotes the norm of the vector. For $f \in L_p^{loc}(\mathbb{R}^n)$, $0 < p \leq \infty$, and any bounded convex domain $\Omega \subset \mathrm{R}^n$ we denote

$$\omega_r(f,\Omega)_p := \omega_r(f, diam(\Omega))_{L_p(\Omega)}. \tag{3.10}$$

Next, for any $\theta \in \Theta$, let $T_{\theta,p} : L_p(\theta) \to \Pi_{r-1}$ be a projector such that

$$\|f - T_{\theta,p} f\|_{L_p(\theta)} \leq c(n,r,p)\, \omega_r(f,\theta)_p, \quad f \in L_p(\theta). \tag{3.11}$$

For $p \geq 1$ the local projectors $T_{\theta,p}$ can be realized as a linear operator using the Averaged Taylor polynomials (see e.g. [9], [10]), but for $0 < p < 1$, $T_{\theta,p}$ are not linear operators. Forming a partition of unity of these local polynomial approximations on each level gives the following operators $T_m : L^p(\mathbb{R}^n) \to \mathcal{S}_m^r$

$$T_m(f) := T_{m,p}(f) := \sum_{\theta \in \Theta_m} T_{\theta,p}(f)\, \tilde{\varphi}_\theta, \quad m \in \mathbb{Z}. \tag{3.12}$$

We record the following properties of these operators

Theorem 3.2 *[6] Let Θ be a discrete cover. Then for any $f \in L_p^{loc}(\mathbb{R}^n)$, $0 < p \leq \infty$,*

(i) $\|T_m f\|_{L_p(\theta)} \leq c \|f\|_{L_p(\theta^*)}$ *, for any $\theta \in \Theta_m$, where $\theta^* := \bigcup_{\theta' \in \Theta_m, \theta \cap \theta' \neq \emptyset} \theta'$.*

(ii) $\|f - T_m f\|_{L_p(\theta)} \leq c \sum_{\theta' \in \Theta_m : \theta' \cap \theta \neq \emptyset} \omega_r(f,\theta')_p$.

(iii) $\|f - T_m f\|_{L_p(\Omega)} \to 0$*, as $m \to \infty$, for any compact $\Omega \subset \mathbb{R}^n$.*

We now proceed with the construction of 'two-level split' representations. Denote

$$\mathcal{M}_m := \{\nu = (\eta,\theta,\beta) : \eta \in \Theta_{m+1}, \theta \in \Theta_m, \eta \cap \theta \neq \emptyset, |\beta| < r\}, \quad m \in \mathbb{Z},$$

and define using (3.2) and (3.5)

$$F_\nu := P_{\eta,\beta} \tilde{\phi}_\eta \tilde{\phi}_\theta = \varphi_{\eta,\beta} \tilde{\phi}_\theta, \quad \nu \in \mathcal{M}_m, \tag{3.13}$$

We also denote

$$\mathcal{F}_m := \{F_\nu : \nu \in \mathcal{M}_m\}, \qquad W_m := span(\mathcal{F}_m).$$

Note that $F_\nu \in C^\infty$, $\mathrm{supp}(F_\nu) = \theta \cap \eta$ if $\nu = (\eta,\theta,\beta)$, and by property (e) in Definition 2.2 we have that $\|F_\nu\|_p \sim |\eta|^{1/p-1/2}$, $0 < p \leq \infty$. It is important that with careful construction, $\mathcal{F}_m$ is a also a stable basis

Theorem 3.3 *[6] If $f \in W_m \cap L_p(\mathrm{R}^n)$, $0 < p \leq \infty$ and $f = \sum_{\nu \in \mathcal{M}_m} a_\nu F_\nu$, then*

$$\|f\|_p \sim \left(\sum_{\nu \in \mathcal{M}_m} \|a_\nu F_\nu\|_p^p \right)^{1/p} \sim 2^{m\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\sum_{\nu \in \mathcal{M}_m} |a_\nu|^p \right)^{1/p}, \tag{3.14}$$

with the obvious modification when $p = \infty$.

Let the coefficients $\left\{A^{\theta,\eta}_{\alpha,\beta}\right\}$ be determined from

$$P_{\theta,\alpha} = \sum_{|\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}, \quad \theta \in \Theta_m, \quad \eta \in \Theta_{m+1}. \tag{3.15}$$

Let $\lambda \in \Lambda_m$ and $\lambda = (\theta, \alpha)$. Then using (3.5), (3.15) and (3.13) we obtain the following *meshless two-scale relationship*

$$\begin{aligned}\varphi_\lambda = P_{\theta,\alpha}\tilde{\phi}_\theta &= \sum_{\eta\in\Theta_{m+1},\, \eta\cap\theta\neq\emptyset} P_{\theta,\alpha}\tilde{\phi}_\theta\tilde{\phi}_\eta = \sum_{\eta\in\Theta_{m+1},\, \eta\cap\theta\neq\emptyset,\, |\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}\tilde{\phi}_\theta\tilde{\phi}_\eta \\ &= \sum_{\eta\in\Theta_{m+1},\, \eta\cap\theta\neq\emptyset,\, |\beta|<r} A^{\theta,\eta}_{\alpha,\beta} F_{\eta,\theta,\beta},\end{aligned}$$

and hence $\varphi_\lambda \in W_m$. Also, if $\lambda \in \Lambda_{m+1}$ and $\lambda = (\eta, \beta)$, then

$$\varphi_\lambda = P_{\eta,\beta}\tilde{\phi}_\eta = \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} P_{\eta,\beta}\tilde{\phi}_\eta\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} F_{\eta,\theta,\beta}.$$

Combining the last two results we find that $\overline{span}\,(\Phi_m \cup \Phi_{m+1}) \subset W_m$.

4 Anisotropic Hardy spaces

Let $\mathcal{S}$ be the *Schwartz class* of rapidly decreasing test functions (in Euclidian sense) and $\mathcal{S}'$ as its dual space. The following definitions are from [8].

Definition 4.1 For $0 < N \leq \widetilde{N}$, let

$$\mathcal{S}_{N,\widetilde{N}} := \left\{\psi \in \mathcal{S} : ||\psi||_{N,\widetilde{N}} := \max_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} (1+|y|)^{\widetilde{N}} |\partial^\alpha \psi(y)| \leq 1\right\}.$$

For a given $\theta(x,t) = x + M_{x,t}(B^*)$ in a continuous cover Θ, we denote

$$\psi_{x,t}(y) := |det(M_{x,t}^{-1})|\psi\left(M_{x,t}^{-1}(x-y)\right). \tag{4.1}$$

Definition 4.2 Let $f \in \mathcal{S}'$ and $\psi \in \mathcal{S}$, and $0 < N \leq \widetilde{N}$. We define, respectively, the *radial maximal function*, and the *grand radial maximal function* of f as

$$M^\circ_\psi f(x) = \sup_{t\in\mathbb{R}} \left| \int f(y)\psi_{x,t}(y)dy \right|,$$

$$M^\circ_{N,\widetilde{N}} f(x) = \sup_{\psi\in\mathcal{S}_{N,\widetilde{N}}} M^\circ_\psi f(x).$$

For a continuous ellipsoid cover Θ of $\mathbb{R}^n$, with parameters $p(\Theta) = (a_1, ..., a_6)$ and $0 < p \leq 1$, define $N_p(\Theta)$ and $\tilde{N}_p(\Theta)$ as the minimal integers satisfying

$$N_p(\Theta) > \frac{max(1, a_4)n + 1}{a_6 p}, \quad \tilde{N}_p(\Theta) > \frac{a_4 N_p(\Theta) + 1}{a_6}. \tag{4.2}$$

Definition 4.3 Let Θ be a continuous ellipsoid cover and let $0 < p \leq 1$. Denoting $M^\circ := M^\circ_{N_p,\tilde{N}_p}$, we define the anisotropic Hardy space as

$$H^p(\Theta) := \{f \in \mathcal{S}' : M^\circ f \in L^p\},$$

with the quasi-norm $||f||_{H^p(\Theta)} := ||M^\circ f||_p$.

Using classical arguments as in Section III of [14], one can show that for any cover Θ and $1 < p \leq \infty$, $H^p(\Theta) \sim L^p(\mathbb{R}^n)$. Therefore we focus our attention to $0 < p \leq 1$. As in the classical case, the anisotropic Hardy spaces can be characterized and then investigated through atomic decompositions.

Definition 4.4 For a cover Θ, we say that (p,q,l) is admissible if $0 \le p \le 1$, $1 \le q \le \infty$, $p < q$, and $l \in \mathbb{N}$, such that $l \ge N_p(\Theta)$ (see (4.2)). A (p,q,l)-atom is a function $a : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $supp(a) \subseteq \theta(x,t)$ for some $\theta(x,t) \in \Theta$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

(ii) $||a||_q \le |\theta(x,t)|^{1/q-1/p}$,

(iii) $\int_{\mathbb{R}^n} a(y) y^\alpha dy = 0$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le l$.

Definition 4.5 Let Θ be an ellipsoid cover, and let (p,q,l) be an admissible triple. We define the atomic Hardy space $H^p_{q,l}(\Theta)$ associated with Θ as the set of all tempered distributions $f \in \mathcal{S}'$ of the form $\Sigma_{i=1}^\infty \lambda_i a_i$, where $\Sigma_{i=1}^\infty |\lambda_i|^p < \infty$ and a_i, $i \ge 1$, is a (p,q,l)-atom. The quasi norm of f is defined as

$$||f||_{H^p_{q,l}(\Theta)} := \inf \left\{ (\Sigma_{i=1}^\infty |\lambda_i|^p)^{1/p} : f = \Sigma_{i=1}^\infty \lambda_i a_i, \;\; a_i \text{ is a } (p,q,l)\text{-atom} \right\}.$$

Theorem 4.6 *[8] For any cover Θ and $0 < p \le 1$, $H^p(\Theta) \sim H^p_{q,l}(\Theta)$ for any (p,q,l) admissible triple.*

5 The dual space of $H^p(\Theta)$

As is known from the general theory of spaces of homogeneous type, the dual of $H^1(\Theta)$ is $BMO(\Theta)$, which for completeness we define here (see [8]). Let Θ be a cover and let $f : \mathbb{R}^n \to \mathbb{R}$. Denote the means over the ellipsoids by

$$f_\theta := \frac{1}{|\theta|} \int_\theta f(x)\, dx, \quad \theta \in \Theta.$$

Then, f is said to belong to the space of *Bounded Mean Oscillation* $BMO(\Theta)$ if there exists a constant $0 < M < \infty$ such that

$$\sup_{\theta \in \Theta} \frac{1}{|\theta|} \int_\theta |f(x) - f_\theta|\, dx \le M.$$

We denote by $\|f\|_{BMO(\Theta)}$ the infimum over all such constants.
Thus, our analysis of dual spaces is focused on the case $0 < p < 1$.

Definition 5.1 Let Θ be an ellipsoid cover (continuous or discrete) over $\mathbb{R}^n$, and let $s \ge 0$, $1 \le q \le \infty$ and $l = 0, 1, 2, ...$ We define the *Campanato*-type space $\mathcal{C}^s_{q,l}(\Theta)$ as the space of functions $g \in L^{loc}_q(\mathbb{R}^n)$ such that

$$||g||_{\mathcal{C}^s_{q,l}(\Theta)} := \sup_{\theta \in \Theta} |\theta|^{-(s+1/q)} \omega_{l+1}(g,\theta)_q < \infty, \tag{5.1}$$

where $\omega_{l+1}(g,\theta)_q$ is the modulus of smoothness of g over θ defined in (3.10).
A few remarks are in order

1. It is known (e.g. [10]) that for $g \in L^{loc}_q(\mathbb{R}^n)$ and any bounded convex domain $\Omega \subset \mathbb{R}^n$, we have the equivalence

$$\inf_{P \in \Pi_l} ||g - P||_{L_q(\Omega)} \sim \omega_{l+1}(f,\Omega)_q, \tag{5.2}$$

where the equivalence constants are independent of g and Ω . This leads us to the following equivalent definition of Campanato spaces. Assume the parameters s, q and l as in Definition 5.1. A function $g \in L^{loc}_q(\mathbb{R}^n)$ belongs to $\mathcal{C}^s_{q,l}(\Theta)$ if

$$\sup_{\theta \in \Theta} \inf_{P \in \Pi_l} |\theta|^{-s} \left(\frac{1}{|\theta|} \int_\theta |g(z) - P(z)|^q dz \right)^{1/q} < \infty, \;\; (q < \infty), \tag{5.3}$$

$$\sup_{\theta \in \Theta} \inf_{P \in \Pi_l} |\theta|^{-s} \operatorname{ess\,sup}_{z \in \theta} |g(z) - P(z)|, \;\; (q = \infty). \tag{5.4}$$

2. By Theorem 2.5, we can replace the ellipsoids in (5.1) by anisotropic balls to get an equivalent norm. In the case where we create a discrete cover by sampling a continuous cover (see Theorem 2.3), where both induce equivalent quasi-distances, this also gives that the Campanato-type spaces constructed over them are equivalent.

3. It is readily seen that $\mathcal{C}^s_{q,l}(\Theta)/\Pi_l$ is a *Banach* space.

The main result of this paper is the following

Theorem 5.2 *Let Θ be a continuous ellipsoid cover and (p,q,l), $0<p<1$, an admissible triple. Then*

$$\left(H^p(\Theta)\right)^* = \left(H^p_{q,l}(\Theta)\right)^* = \mathcal{C}^{1/p-1}_{q',l}(\Theta)/\Pi_l \text{ where } 1/q'+1/q=1. \tag{5.5}$$

For the proof of Theorem 5.2 we need to follow the careful analysis of [1] and [3], but adapted to our highly anisotropic setting. Although it is easy to prove uniform boundedness of the action of $g\in\mathcal{C}^s_{q',l}(\Theta)$ on $H^p(\Theta)$ atoms, this does not automatically guarantee boundedness on $H^p(\Theta)$. For example, in general, one cannot assume that $l(f)=\sum_i \lambda_i l(a_i)$ holds for a linear functional l that is uniformly bounded on atoms and an atomic representation $f=\sum_i\lambda_i a_i$ (see the detailed discussion and counter-examples for the isotropic case in [2]). Thus, we shall use an approximation argument, through a series of Lemmas.

The approximation argument proceeds as follows. First, Lemma 5.3 shows that for the stronger case where $g\in\mathcal{S}$ and any $f\in H^p(\Theta)$ such that $f=\sum_i\lambda_i a_i$, with $\sum_i|\lambda_i|^p<\infty$, where a_i are (p,q,l)-atoms for an admissible triplet, we do have

$$\int fg = \sum_i \lambda_i \int a_i g. \tag{5.6}$$

Then, in Lemma 5.5 we generalize this (in some weaker sense) for $g\in\mathcal{C}^{1/p-1}_{q',l}(\Theta)\cap C^\infty$, through an approximation of the action of g by a sequence of Schwartz functions. Finally Lemma 5.6 shows equality (5.6) for any $g\in\mathcal{C}^{1/p-1}_{q',l}(\Theta)$, through the approximations $T_m g$, where T_m, $m\in\mathbb{Z}$, are the operators (3.12) mapping $L_{q'}$ to C^∞. Once the technical goal of validating (5.6) for arbitrary $g\in\mathcal{C}^{1/p-1}_{q',l}(\Theta)$ is achieved, we can proceed with the proof of Theorem 5.2.

Lemma 5.3 *Let (p,q,l) be an admissible triple, and $f=\sum_i\lambda_i a_i\in H^p_{q,l}(\Theta)$, with $\sum_i|\lambda_i|^p<\infty$. Then*

$$\int f\psi = \sum_i \lambda_i \int a_i\psi, \ \ \forall\psi\in\mathcal{S}. \tag{5.7}$$

Proof. For $\psi\in\mathcal{S}$ and $x\in\theta(0,0)$, denote $\psi^x(y) := |M_{x,0}|\,\psi(-M_{x,0}y+x)$. Since by (2.2) for the points $x\in\theta(0,0)$, all the ellipsoids $\theta(x,0)$ have 'equivalent' shape, it is not difficult to see that there exists a constant $c(\|M_{0,0}\|, p(\Theta))$, such that

$$\|\psi^x\|_{N,\tilde{N}} \le c\,\|\psi\|_{N,\tilde{N}}, \quad \forall x\in\theta(0,0).$$

Observe that (using the notation (4.1))

$$|\langle f,\psi\rangle| = \left|\int_{\mathbb{R}^n} f\psi^x_{x,0}\right| \le c\,\|\psi\|_{N,\tilde{N}}\, M^\circ f(x).$$

Therefore

$$|\langle f,\psi\rangle|^p \le c\int_{\theta(0,0)} M^\circ f(x)^p\,dx \le c\,\|f\|^p_{H^p(\Theta)}, \tag{5.8}$$

where the constant depends on the cover and ψ. Now, assume $f=\sum_i\lambda_i a_i$ is an atomic decomposition of $f\in H^p(\Theta)$, with $\sum_i|\lambda_i|^p<\infty$. Then

$$|\langle f-\textstyle\sum_{i=1}^m\lambda_i a_i,\psi\rangle| \le c\,\|f-\textstyle\sum_{i=1}^m\lambda_i a_i\|_{H^p(\Theta)} \to 0 \text{ as } m\to\infty,$$

which shows (5.7). □

For the next Lemma, we recall ([6], [8]) that there exists a constant $\gamma(p(\Theta))$ such that for all $x,y\in\mathbb{R}^n$ and $t_1,t_2\in\mathbb{R}$, if $\theta(x,t_1)\cap\theta(y,t_2)\neq\emptyset$ with $t_1\le t_2$, then $\theta(y,t_2)\subset\theta(x,t_1-\gamma)$.

Lemma 5.4 *Let Θ be an ellipsoid cover such that $B^* = \theta(0,0) \in \Theta$. Let ϕ be a smooth function such that supp(ϕ) $\subset \theta(0,\gamma)$, $0 \le \phi(x) \le 1$ for all $x \in \mathbb{R}^n$, and $\phi(x) = 1$ for all $x \in \theta(0,2\gamma)$. Then, for any integer $l' \ge l + \frac{s+1/q'}{a_6} - 1$ there exists $c > 0$ such that for any $g \in \mathcal{C}^s_{q',l}(\Theta)$ one has*

$$\|(g - T_{B^*,q'}g)\phi\|_{\mathcal{C}^s_{q',l'}(\Theta)} \le c\,\|g\|_{\mathcal{C}^s_{q',l}(\Theta)}\,,$$

where $T_{\theta,q}$ are the linear polynomial approximations satisfying (3.11), with $r - 1 = l$.

Proof. Consider $g \in \mathcal{C}^s_{q',l}(\Theta)$ with $||g||_{\mathcal{C}^s_{q',l}(\Theta)} \le 1$, denote $\widetilde{g} := (g - T_{B^*,q'}g)\phi$ and let $\theta = \theta(x,t) \in \Theta$. We have two cases to consider. For the first case, $t \le \gamma$, we have

$$\inf_{P \in P_{l'}} |\theta|^{-s} \left(\tfrac{1}{|\theta|}\int_\theta |\widetilde{g} - P|^{q'}\right)^{1/q'}$$

$$\le |\theta|^{-s}\left(\tfrac{1}{|\theta|}\int_{\mathbb{R}^n} |\widetilde{g}|^{q'}\right)^{1/q'}$$

$$\le |\theta|^{-(s+1/q')}\left(\int_{B^*} |g - T_{B^*,q'}g|^{q'}\right)^{1/q'}$$

$$\le c|\theta|^{-(s+1/q')}\omega_{l+1}(g,B^*)_{q'} \le c||g||_{\mathcal{C}^s_{q',l}(\Theta)}.$$

We now prove the second case, $t \ge \gamma$. Since the support of $\widetilde{g}$ is contained in $\theta(0,\gamma)$, and since we want to estimate its Campanato norm, we may consider only ellipsoids $\theta(x,t)$ at levels $\ge \gamma$, which intersect $\theta(0,\gamma)$, and therefore are contained in B^*. We denote $G := g - T_{B^*,q'}g$ and $P_1 := T_{\theta,q'}G$. We have

$$\|P_1\|_{L_{q'}(\theta)} \le c\,\|G\|_{L_{q'}(\theta)}$$
$$\le c\,\|g - T_{B^*,q'}g\|_{L_{q'}(B^*)}$$
$$\le c\,\|g\|_{\mathcal{C}^s_{q,l}(\Theta)} \le c,$$

and

$$\|G - P_1\|_{L_{q'}(\theta)} = \|g - T_{\theta,q'}g\|_{L_{q'}(\theta)} \le c\,|\theta|^{s+1/q'}\,.$$

Let P_2 be the *Taylor polynomial* of ϕ around x of degree $l'' \ge \frac{s+1/q'}{a_6} - 1$. Since $\theta(x,t) \cap B^* \ne \emptyset$ and $t > 0$, (2.2) implies that $\mathrm{diam}(\theta) \le c2^{-a_6 t}$. Therefore, the Taylor Remainder Theorem gives

$$\sup_{z\in\theta} |\phi(z) - P_2(z)| \le C|x - z|^{l''+1}$$

$$\le c(2^{-t})^{(l''+1)a_6} \le C(2^{-t})^{s+1/q'}$$

$$\le c|\theta|^{s+1/q'}.$$

Consider the polynomial $P := P_1P_2$ of degree $\le \deg(P_1) + \deg(P_2) \le l + l'' := l'$. Applying the last three estimates we conclude

$$\|\tilde{g} - P_1P_2\|_{L_{q'}(\theta)} = \|G\phi - P_1P_2\|_{L_{q'}(\theta)}$$

$$\le \|G\phi - P_1\phi\|_{L_{q'}(\theta)} + \|P_1\phi - P_1P_2\|_{L_{q'}(\theta)}$$

$$\le \|G - P_1\|_{L_{q'}(\theta)} + \sup_{z\in\theta}|\phi\,(z) - P_2\,(z)|\,\|P_1\|_{L_{q'}(\theta)}$$

$$\le c\,|\theta|^{s+1/q'}\,.$$

□

For an admissible triplet (p,q,l), $0<p<1$, we denote by $\mathcal{M}_l^q$ as the set of all q integrable functions, that have compact support and l vanishing moments, namely

$$\mathcal{M}_l^q := \{f \in L^q(\mathbb{R}^n) : f \text{ has compact support, } \int f(x)x^\alpha = 0,\ \forall |\alpha| \le l\}.$$

Note that $\mathcal{M}_l^q \subset H^p(\Theta)$, for any cover Θ, since for any $f \in \mathcal{M}_l^q$ whose support is contained in an ellipsoid $\theta \in \Theta$, the function $|\theta|^{(1/q-1/p)}||f||_q^{-1}f$, is in fact an atom. Moreover, $\mathcal{M}_l^q$ is dense in $H^p(\Theta)$.

Lemma 5.5 *Suppose $g \in \mathcal{C}^s_{q',l}(\Theta) \cap C^\infty$, where $s \ge 0$, $1 \le q' \le \infty$, $l = 0,1,2,...$ Let l' be an integer satisfying $l' \ge l + \frac{s+1/q'}{a_6} - 1$. Then there exists and a constant $c > 0$ independent of g, and a sequence $\{g_m\}_m \subset \mathcal{S}$ such that*

$$||g_m||_{\mathcal{C}^s_{q',l'}(\Theta)} \le c||g||_{\mathcal{C}^s_{q',l}(\Theta)} \quad \forall m \in \mathbb{N}, \tag{5.9}$$

$$\int g_m f \to \int gf,\ \text{as } m \to \infty,\ \forall f \in \mathcal{M}_l^q. \tag{5.10}$$

Proof. For every $m \in \mathbb{N}$, let M_m be the matrix associated with $\theta_m := \theta(0,-m)$, such that $M_m(B^*) = \theta_m$. Consider $\phi \in C^\infty$ as in Lemma 5.4. We denote $g'_m := g(M_m\cdot)$, and $g''_m := \big(g'_m - T_{B^*,q'}g'_m\big)\phi$. We define the sequence $\{g_m\}_m$ by

$$g_m := g''_m(M_m^{-1}\cdot).$$

It is not hard to see $g_m \in C^\infty$, and has a compact support. Moreover, condition (5.10) follows immediately from the construction of $\{g_m\}_m$. Thus, it remains to show condition (5.9). In what follows, we use the fact that for every invertible matrix M and $0 < q \le \infty$,

$$\omega_r\big(f(M\cdot), M^{-1}\theta\big)_q = |\det M|^{-1/q}\omega_r(f,\theta)_q. \tag{5.11}$$

For every $m \in \mathbb{N}$, consider the ellipsoid cover $\widetilde{\Theta} := M_m^{-1}\Theta$. It is obvious that $B^* = M_m^{-1}(\theta_m) \in \widetilde{\Theta}$. For any $f \in \mathcal{C}^s_{q',l}(\Theta)$, denote $\tilde{f} := f(M_m\cdot)$. By our construction and (5.11)

$$\begin{aligned}
\left\|\tilde{f}\right\|_{\mathcal{C}^s_{q',l}(\widetilde{\Theta})} &:= \sup_{\tilde{\theta}\in\widetilde{\Theta}} |\tilde{\theta}|^{-(s+1/q')}\omega_{l+1}(\tilde{f},\tilde{\theta})_{q'} \\
&= \sup_{\theta\in\Theta} \left|M_m^{-1}(\theta)\right|^{-(s+1/q')}\omega_{l+1}\big(f(M_m\cdot), M_m^{-1}(\theta)\big)_{q'} \\
&= |\det M_m|^{s+1/q'} \sup_{\theta\in\Theta} |\theta|^{-(s+1/q')}\omega_{l+1}\big(f(M_m\cdot), M_m^{-1}(\theta)\big)_{q'} \\
&= |\det M_m|^{s+1/q'} \sup_{\theta\in\Theta} |\theta|^{-(s+1/q')}|\det M_m|^{-1/q'}\omega_{l+1}\big(f,\theta\big)_{q'} \\
&= |\det M_m|^{s} \sup_{\theta\in\Theta} |\theta|^{-(s+1/q')}\omega_{l+1}\big(f,\theta\big)_{q'} = |\det M_m|^s \|f\|_{\mathcal{C}^s_{q',l}(\Theta)}.
\end{aligned}$$

Since $g \in \mathcal{C}^s_{q',l}(\Theta) \cap C^\infty$ we conclude $g'_m := g(M_m\cdot) \in \mathcal{C}^s_{q',l}\big(\widetilde{\Theta}\big) \cap C^\infty$. Moreover $\|g'_m\|_{\mathcal{C}^s_{q',l}(\widetilde{\Theta})} \le |\det M_m|^s \|g\|_{\mathcal{C}^s_{q',l}(\Theta)}$. As $B^* \in \widetilde{\Theta}$, Lemma 5.4 implies

$$\begin{aligned}
\|g''_m\|_{\mathcal{C}^s_{q',l'}(\widetilde{\Theta})} &= \left\|\big(g'_m - T_{B^*,q'}g'_m\big)\phi\right\|_{\mathcal{C}^s_{q',l'}(\widetilde{\Theta})} \\
&\le c\,\|g'_m\|_{\mathcal{C}^s_{q',l}(\widetilde{\Theta})} \\
&\le c|\det M_m|^s \|g\|_{\mathcal{C}^s_{q',l}(\Theta)}.
\end{aligned} \tag{5.12}$$

We then have by the construction, (5.11) and (5.12)

$$||g_m||_{\mathcal{C}^s_{q',l'}(\Theta)} := ||g''_m(M_m^{-1}\cdot)||_{\mathcal{C}^s_{q',l'}(\Theta)}$$

$$= \sup_{\theta\in\Theta} |\theta|^{-(s+1/q')}\omega_{l'+1}\big(g''_m(M_m^{-1}\cdot),\theta\big)_{q'}$$

$$= |\det M_m^{-1}|^s \sup_{\widetilde{\theta}\in\widetilde{\Theta}} |\widetilde{\theta}|^{-(s+1/q')}\omega_{l'+1}\big(g''_m,\widetilde{\theta}\big)_{q'}$$

$$= |\det M_m^{-1}|^s \, \|g''_m\|_{\mathcal{C}^s_{q',l'}(\widetilde{\Theta})} \le c\,\|g\|_{\mathcal{C}^s_{q',l}(\Theta)}\,.$$

□

Lemma 5.6 *If $g \in \mathcal{C}^s_{q',l}(\Theta)$, where $s \ge 0$, $1 \le q' \le \infty$, $a_6(l+1) > s$, then*

$$||T_m g||_{\mathcal{C}^s_{q',l}(\Theta)} \le c||g||_{\mathcal{C}^s_{q',l}(\Theta)} \quad \forall m \in \mathbb{Z}, \tag{5.13}$$

$$\int T_m(g)f \to \int gf, \text{ as } m \to \infty, \ \forall f \in \mathcal{M}^q_l, \tag{5.14}$$

where $T_m = T_{m,q'}$, $m \in \mathbb{Z}$, are the operators (3.12) defined over the cover Θ if it is discrete, or over a 'discretization' of a continuous cover per Theorem 2.3.

Proof. Without loss of generality Θ is a discrete cover and the operators T_m are well defined over it. For any $f \in \mathcal{M}^q_l$, by the relation $1/q' + 1/q = 1$, Hölder inequality and Theorem 3.2 (iii) we get

$$\left|\int_{\mathbb{R}^n} fT_m g - \int_{\mathbb{R}^n} fg\right| \le \int_{\text{supp}(f)} |f(T_m g - g)|$$

$$\le ||f||_{L^q(\mathbb{R}^n)} \left(\int_{\text{supp}(f)} |T_m g - g|^{q'}\right)^{1/q'} \to 0 \text{ as } m \to \infty.$$

Thus, it remains to show condition (5.13), namely, the uniform boundedness of $T_j g$, $j \in \mathbb{Z}$, with respect the Campanato norm. To this end, let $j \in \mathbb{Z}$ and let $\theta(x,m) \in \Theta_m$. We need to estimate $|\theta|^{-(s+1/q')}\omega_{l+1}(T_j g,\theta)_{q'}$ using $||g||_{\mathcal{C}^s_{q',l}(\Theta)}$. We first prove the case where $m \le j$. Assume momentarily there exists $\eta \in \Theta_{m-J}$ with $J > 0$, independent of g, j, m and θ such that

$$\omega_{l+1}\big(T_j g - g,\theta\big)_{q'} \le c\omega_{l+1}\big(g,\eta\big)_{q'}. \tag{5.15}$$

We then get

$$2^{m(s+1/q')}\omega_{l+1}\big(T_j g,\theta\big)_{q'}$$

$$\le 2^{m(s+1/q')}\omega_{l+1}\big(T_j g - g,\theta\big)_{q'} + 2^{m(s+1/q')}\omega_{l+1}\big(g,\theta\big)_{q'}.$$

$$\le c2^{(m-J)(s+1/q')}\omega_{l+1}\big(g,\eta\big)_{q'} + 2^{m(s+1/q')}\omega_{l+1}\big(g,\theta\big)_{q'}$$

$$\le c||g||_{\mathcal{C}^s_{q',l}(\Theta)}.$$

Thus, to complete the proof of the first case one should prove (5.15). Consider the set $\Gamma'_\theta := \{\theta' \in \Theta_j : \theta' \cap \theta \ne \emptyset\}$ and then the set $\Gamma''_\theta := \{\theta'' \in \Theta_j : \theta'' \cap \theta' \ne \emptyset$ for some $\theta' \in \Gamma'_\theta\}$. By the properties of ellipsoid covers, there exists $\eta \in \Theta_{m-J}$, for $J(p(\Theta)) > 0$, such that $\theta'' \subset \eta$, for each $\theta'' \in \Gamma''_\theta$. Consequently, by Theorem 3.2 (ii) and

(5.2)

$$\omega_{l+1}\big(g-T_jg,\theta\big)_{q'}^{q'} \le c||g-T_jg||_{L^{q'}(\theta)}^{q'}$$

$$\le c\sum_{\theta'\in\Gamma'_\theta}||g-T_jg||_{L^{q'}(\theta')}^{q'}$$

$$\le c\sum_{\theta''\in\Gamma''_\theta}\omega_{l+1}(g,\theta'')_{q'}^{q'}$$

$$\le c\omega_{l+1}\big(g,\eta\big)_{q'}^{q'}.$$

For the $j<m$ case, we apply a telescopic sum argument

$$\omega_{l+1}\big(T_jg,\theta\big)_{q'} \le \sum_{k=j}^{m-1}\omega_{l+1}\big((T_k-T_{k+1})g,\theta\big)_{q'}+\omega_{l+1}(T_mg,\theta)_{q'}.$$

Assume momentarily that for $\alpha := a_6(l+1)-s>0$

$$2^{m(s+1/q')}\omega_{l+1}\big((T_k-T_{k+1})g,\theta\big)_{q'} \le c2^{(k-m)\alpha}||g||_{\mathcal{C}^s_{q',l}(\Theta)}. \tag{5.16}$$

Then

$$2^{m(s+1/q')}\omega_{l+1}\big(T_jg,\theta\big)_{q'}$$

$$\le \sum_{k=j}^{m-1}2^{m(s+1/q')}\omega_{l+1}\big((T_k-T_{k+1})g,\theta\big)_{q'}+2^{m(s+1/q')}\omega_{l+1}(T_mg,\theta)_{q'}$$

$$\le c\left(\sum_{k=j}^{m-1}2^{(k-m)\alpha}\right)||g||_{\mathcal{C}^s_{q',l}(\Theta)}+c||g||_{\mathcal{C}^s_{q',l}(\Theta)} \le c||g||_{\mathcal{C}^s_{q',l}(\Theta)}.$$

To prove (5.16), we use the 'Two Level Split' representation at the level k (see (3.13)) over θ

$$(T_k-T_{k+1})g\big|_\theta = \sum_{\nu\in\mathcal{M}_k,\ \eta_\nu\cap\theta\neq\emptyset} a_\nu F_\nu.$$

From (A.4) in [6] for $\theta\in\Theta_m$, $F_\nu\in\mathcal{F}_k$, $k\le m$, such that $\eta_\nu\cap\theta\neq\emptyset$ one has

$$\omega_{l+1}\big(F_\nu,\theta\big)_{q'}^{q'} \le c\big(|\theta|/|\eta_\nu|\big)2^{-a_6(l+1)(m-k)q'}||F_\nu||_{q'}^{q'}$$

$$\le c2^{(k-m)\left(1/q'+a_6(l+1)\right)q'}||F_\nu||_{q'}^{q'}.$$

Let $\Omega_\theta := \cup_{\eta_\nu\in\Theta_{k+1},\eta_\nu\cap\theta\neq\emptyset}\eta_\nu$. Since $k<m$, only a bounded number of ellipsoids η_ν at the level $k+1$ intersect θ. This together with Theorem 3.3 yield

$$\omega_{l+1}\big((T_k-T_{k+1})g,\theta\big)_{q'}^{q'} \le c\sum_{\nu\in\mathcal{M}_k,\ \eta_\nu\cap\theta\neq\emptyset}\omega_{l+1}\big(a_\nu F_\nu,\theta\big)_{q'}^{q'}$$

$$\le c2^{(k-m)\left(1/q'+a_6(l+1)\right)q'}\sum_{\nu\in\mathcal{M}_k,\ \eta_\nu\cap\theta\neq\emptyset}||a_\nu F_\nu||_{q'}^{q'}$$

$$\le c2^{(k-m)\left(1/q'+a_6(l+1)\right)q'}||(T_k-T_{k+1})g||_{L^{q'}(\Omega_\theta)}^{q'}.$$

As before, there exists a constant $J(p(\Theta))>0$ and an ellipsoid $\eta'\in\Theta_{k-J}$ that contains all the ellipsoids from levels $k,k+1$, that intersect with Ω_θ. Therefore, by Theorem 3.2

$$||(T_k-T_{k+1})g||_{L^{q'}(\Omega_\theta)} \le \|T_kg-g\|_{L^{q'}(\Omega_\theta)}+\|g-T_{k+1}g\|_{L^{q'}(\Omega_\theta)}$$

$$\le c\omega_{l+1}(g,\eta')_{q'}.$$

We may now conclude (5.16) by

$$2^{m(s+1/q')}\omega_{l+1}\big((T_k - T_{k+1})g,\theta\big)_{q'} \le c2^{m(s+1/q')}2^{(k-m)\left(1/q'+a_6(l+1)\right)}\omega_{l+1}(g,\eta')_{q'}$$

$$\le c2^{(k-m)(-s+a_6(l+1))}2^{(k-J)(s+1/q')}\omega_{l+1}(g,\eta')_{q'}$$

$$\le c2^{(k-m)\alpha}||g||_{\mathcal{C}^s_{q',l}(\Theta)}.$$

□

Lemma 5.7 *Let (p,q,l') be an admissible triple, where $0 < p < 1 \le q$. Then, for any $g \in \mathcal{C}^{1/p-1}_{q',l'}(\Theta)$ and any (p,q,l')-atom a*

$$\left|\int ga\right| \le c||g||_{\mathcal{C}^{1/p-1}_{q',l'}(\Theta)} \tag{5.17}$$

Proof. For a (p,q,l')-atom a associated with an ellipsoid θ, we have

$$\left|\int ga\right| = \inf_{P\in\Pi_{l'}}\left|\int(g-P)a\right|$$

$$\le \left(\inf_{P\in\Pi_{l'}}\int_\theta |g-P|^{q'}\right)^{1/q'}||a||_q$$

$$\le \left(\inf_{P\in\Pi_{l'}}\int_\theta |g-P|^{q'}\right)^{1/q'}|\theta|^{1/q-1/p} \le c||g||_{\mathcal{C}^{1/p-1}_{q',l'}(\Theta)}.$$

□

We can now apply the approximation arguments to conclude

Corollary 5.8 *Let (p,q,l) be an admissible triple, where $0 < p < 1 \le q$. Then, for any $g \in \mathcal{C}^{1/p-1}_{q',l}(\Theta)$, $f \in \mathcal{M}^q_l$, $f = \sum_i \lambda_i a_i$, where $\sum_i |\lambda_i|^p < \infty$, and a_i are (p,q,l')-atoms, $l' \ge l + \frac{1/p+1/q'-1}{a_6} - 1$, then*

$$\int fg = \sum_i \lambda_i \int ga_i. \tag{5.18}$$

Proof. The admissibility of the triplet (p,q,l) implies by (4.2) that $a_6(l+1) > 1/p-1$. Therefore, the conditions of Lemma 5.6 are fulfilled and $\{T_m g\}$ is a sequence of C^∞ functions which satisfies (5.13) and (5.14). We claim that it is sufficient to prove (5.18) for $g \in \mathcal{C}^{1/p-1}_{q',l}(\Theta) \cap C^\infty$. Indeed, if (5.18) holds for the sequence $\{T_m g\}$, we proceed as follows to obtain (5.18) for g. Observe that (5.13) and (5.17) provide a uniform bound for all i, m

$$\left|\int a_i T_m g\right| \le c||g||_{\mathcal{C}^{1/p-1}_{q',l}(\Theta)}.$$

An additional uniform bound for all m is

$$\sum_i |\lambda_i|\left|\int a_i T_m g\right| \le c||g||_{\mathcal{C}^{1/p-1}_{q',l}(\Theta)}(\sum_i |\lambda_i|^p)^{1/p}.$$

Thus, we may invoke (5.14), (5.18) for the sequence $\{T_m g\}$ and then Lebesgue's Dominated Convergence theorem applied to the counting measure on $\mathbb{N}$ to obtain

$$\int fg = \lim_{m\to\infty}\int fT_m g = \lim_{m\to\infty}\sum_i \lambda_i \int a_i T_m g = \sum_i \lambda_i \int ga_i.$$

Next, applying Lemma 5.5 and using the assumption that the atomic representation is of degree $l' \geq l + \frac{1/p+1/q'-1}{a_6} - 1$, we can repeat the preceding argument and further assume that $g \in \mathcal{C}^{1/p-1}_{q',l'}(\Theta) \cap \mathcal{S}$. Finally, Lemma 5.3 proves that for $g \in \mathcal{S}$ and $f \in H^p(\Theta)$, (5.18) holds. □

As we already discussed, a linear functional that is uniformly bounded on atoms, may not be bounded on the Hardy space [2]. Yet, if the functional is a priori known to be bounded, then its norm may be determined from the bound on atoms.

Lemma 5.9 *Let F be a linear continuous functional on $H^p_{q,l}(\Theta)$, where (p,q,l) is an admissible triple. Then*

$$||F||_{(H^p_{q,l}(\Theta))^*} := \sup\{|Ff| : ||f||_{H^p_{q,l}(\Theta)} \leq 1\} = \sup\{|Fa| : a \text{ is a } (p,q,l)-\text{atom}\}.$$

Proof. By definition 4.5, for every (p,q,l)-atom a, we have $||a||_{H^p_{q,l}(\Theta)} \leq 1$. Thus,

$$\sup\{|Fa| : a \text{ is a } (p,q,l)-\text{atom}\} \leq \sup\{|Ff| : ||f||_{H^p_{q,l}(\Theta)} \leq 1\}.$$

In the other direction, consider $f \in H^p(\Theta)$ such that $||f||_{H^p_{q,l}(\Theta)} \leq 1$. Then, for every $\varepsilon > 0$, there exists an atomic representation $f = \sum_i \lambda_i a_i$, in the sense of $H^p_{q,l}(\Theta)$,such that $\left(\sum_i |\lambda_i|^p\right)^{1/p} < 1 + \varepsilon$. Since F is a bounded linear functional, $Ff = \sum_i \lambda_i F a_i$ and therefore

$$\begin{aligned}
&|Ff| \leq \textstyle\sum_i |\lambda_i||Fa_i| \\
&\leq \left(\textstyle\sum_i |\lambda_i|^p\right)^{1/p} \sup\{|Fa| : a \text{ is a } (p,q,l)-\text{atom}\} \\
&\leq (1+\varepsilon)\sup\{|Fa| : a \text{ is a } (p,q,l)-\text{atom}\}.
\end{aligned}$$

□

We are now ready to prove our main result.

Proof of Theorem 5.2. We begin with $\left(H^p(\Theta)\right)^* \subseteq \mathcal{C}^{1/p-1}_{q',l}(\Theta)$, where (p,q,l) are an admissible triplet and q' is the conjugate index of q. To this end we prove that for any linear functional $F_g \in \left(H^p(\Theta)\right)^*$, there exists $g \in L^{loc}_{q'}(\mathbb{R}^n)$ such that for any $f \in H^p(\Theta)$

$$F_g f = \int fg \qquad \text{and} \qquad ||g||_{\mathcal{C}^{1/p-1}_{q',l}(\Theta)} \leq c||F_g||_{(H^p(\Theta))^*}.$$

We prove the case $0 < p < 1 \leq q < \infty$ (see [1] for more details and the case $q = \infty$). For any $\theta \in \Theta$, let $L^0_q(\theta) := \{f \in L_q(\theta) : T_{\theta,q} f = 0\}$, where $T_{\theta,q}$ is the polynomial approximation operator of degree l. Here, we assume f vanishes outside of θ and therefore one can identify its normalized version, $|\theta|^{1/q-1/p}\|f\|_q^{-1} f$ as an (p,q,l)-atom, with $H^p_{q,l}(\Theta)$-norm ≤ 1. Consequently, since F_g is assumed to be a priori a bounded operator, for all $f \in L^0_q(\theta)$

$$|F_g f| \leq \|F_g\|_{(H^p_{q,l}(\Theta))^*} |\theta|^{1/p-1/q} \|f\|_q. \tag{5.19}$$

Recall that there exists $J(p(\Theta)) > 0$, such that $\theta(x,t) \subset \theta(x, t-J)$, for any $x \in \mathbb{R}^n, t \in \mathbb{R}$. By (5.19), for any $m \geq 0$ and $f \in L^0_q(\theta(0,-Jm))$, we have that

$$|F_g f| \leq \|F_g\|_{(H^p_{q,l}(\Theta))^*} |\theta(0,-Jm)|^{1/p-1/q} \|f\|_q.$$

By the Hahn-Banach Theorem, F_g can be extended to the space $L_q(\theta(0,-Jm))$ without increasing its norm. By duality, there exists a unique function $g_m \in L_{q'}(\theta(0,-Jm))$ (up to a set of measure zero and a polynomial of degree l), such that $F_g f = \int_{\theta(0,-Jm)} g_m f$ for all $f \in L^0_q(\theta(0,-Jm))$. It is readily seen that that $g_{m+1}|_{\theta(0,-Jm)} = g_m$ and consequently one may identify the action of the functional F_g with a function

$g \in L_{q'}^{loc}(\mathbb{R}^n)$, by setting $g(x) = g_m(x)$, if $x \in \theta(0, -Jm)$. By (5.19), for any $\theta \in \Theta$, the norm of g as a functional on $L_q^0(\theta)$, satisfies

$$\|g\|_{L_q^0(\theta)^*} \le \|F_g\|_{(H_{q,l}^p(\Theta))^*} |\theta|^{1/p-1/q}.$$

Since $L_q^0(\theta)^* = L_{q'}(\theta)/\Pi_l$, we have that $\|g\|_{L_q^0(\theta)^*} = \inf_{P\in\Pi_l} \|g-P\|_{L_{q'}(\theta)}$. We may now conclude that

$$\begin{aligned}
&\|g\|_{\mathcal{C}_{q',l}^{1/p-1}(\Theta)} = \sup_{\theta\in\Theta} |\theta|^{-(1/p-1+1/q')} \omega_{l+1}(g,\theta)_{q'} \\
&\le c \sup_{\theta\in\Theta} \left(|\theta|^{1/q-1/p} \inf_{P\in\Pi_l} \|g-P\|_{L_{q'}(\theta)}\right) \\
&\le c \|F_g\|_{(H^p(\Theta))^*}.
\end{aligned}$$

We now prove the second direction. Let $g \in \mathcal{C}_{q',l}^{1/p-1}(\Theta)$ and denote $F_g f := \int fg, \ \forall f \in \mathcal{M}_l^q$. Let $f = \sum_i \lambda_i a_i$ be an atomic decomposition, such that a_i are (p,q,l') atoms, $l' \ge l + \frac{1/p+1/q'-1}{a_6} - 1$ and $\left(\sum_i |\lambda_i|^p\right)^{1/p} \le 2||f||_{H_{q,l'}^p(\Theta)}$. Corollary 5.8 and then Lemma 5.7 yield

$$\begin{aligned}
&|F_g f| = |\sum_i \lambda_i F_g a_i| \le \sum_i |\lambda_i||F_g a_i| \\
&\le c||g||_{\mathcal{C}_{q',l}^{1/p-1}(\Theta)} \left(\sum_i |\lambda_i|^p\right)^{1/p} \\
&\le c||g||_{\mathcal{C}_{q',l}^{1/p-1}(\Theta)} ||f||_{H_{q,l'}^p(\Theta)} \\
&\le c||g||_{\mathcal{C}_{q',l}^{1/p-1}(\Theta)} ||f||_{H^p(\Theta)}.
\end{aligned}$$

Next, using the density of $\mathcal{M}_l^q$ in $H^p(\Theta)$, we extend F_g (uniquely) to a bounded functional on $H^p(\Theta)$. But since by Lemma 5.9 the norm of a bounded functional is determined by its action on atoms, we have that $|F_g f| \le c||g||_{\mathcal{C}_{q',l}^{1/p-1}(\Theta)} ||f||_{H^p(\Theta)}$, for all $f \in H^p(\Theta)$.

□

Remark It is not known yet if the proof of the continuous embedding $\mathcal{C}_{q',l}^{1/p-1}(\Theta) \subseteq \left(H^p(\Theta)\right)^*$, can be simplified by defining the functional F_g, $g \in \mathcal{C}_{q',l}^{1/p-1}(\Theta)$, on finite sums of atoms and then using a density argument. This is because, in general, one cannot bound the l_p norm of the coefficients of a finite sum of atoms by its atomic Hardy norm, since the atomic Hardy norm is possibly achieved using the infimum over infinite sums (of other atoms) [1]. However, in some special cases this is known to be true [13].

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