Mathematical Foundations of ML – Function Spaces I

Banach space = complete normed vector space B over a field $F=\{\mathbb{R},\mathbb{C}\}$,

$$x,y\in B,\ \alpha,\beta\in F \Rightarrow \alpha x+\beta y\in B.$$

i. $f\neq 0 \Rightarrow \|f\|>0$,

ii. $\|\alpha f\|=|\alpha|\|f\|$,

iii. Triangle inequality $\|f+g\|\leq\|f\|+\|g\|$.

Measure

In this course we only use the standard Lebesgue measure – the volume of a (measurable) set.

We will need the notation of zero measure (volume). Example: a set of discrete points

Radon measure – compatible with topology of space

i. σ -measurable on Borel sets,

ii. locally finite (every point has a neighborhood of finite measure),

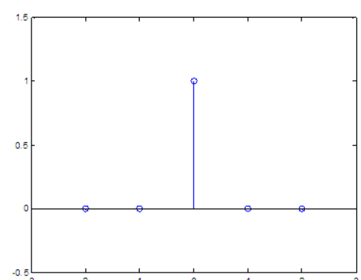
iii. inner regular (measure of a set can be approximated by measure of compact sets

Lp Spaces

$\Omega\subseteq\mathbb{R}^n$ domain. Examples: $\Omega=[a,b]\subset\mathbb{R},\ \Omega=[0,1]^n\subset\mathbb{R}^n, \Omega=\mathbb{R}^n$.

$$\|f\|_{L_p(\Omega)}:=\begin{cases}\left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}, & 0<p<\infty,\\ ess\sup_{x\in\Omega}|f(x)|, & p=\infty.\end{cases}$$

$$ess\sup_x |f(x)|:=\sup_{A>0}\left\{A>0:\ \left|\{x: |f(x)|\geq A\}\right|>0\right\}.$$



$1\leq p\leq\infty$ Banach spaces

$0<p<1$ Quasi-Banach spaces (quasi-triangle inequality holds)

Hölder $1\leq p\leq\infty$, $f\in L_p, g\in L_{p'}$

$$\|fg\|_1\leq\|f\|_p\|g\|_{p'} \qquad \frac{1}{p}+\frac{1}{p'}=1.$$

Schwartz $p=2$

$$\langle f,g\rangle_2=\left|\int_\Omega f\bar{g}\right|\leq\|fg\|_1=\int_\Omega|fg|\leq\|f\|_2\|g\|_2.$$

Lp spaces not comparable on unbounded domains

Example We'll use $\Omega = \mathbb{R}$. Assume $0 < q < p < \infty$
Choose

$$f(x) := \begin{cases} 0 & |x| \le 1 \\ \dfrac{1}{|x|^{1/q}} & |x| > 1 \end{cases}$$

We have $f \in L_p(\mathbb{R})$, $f \notin L_q(\mathbb{R})$
Now choose

$$f(x) := \begin{cases} \dfrac{1}{|x|^{1/p}} & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

We have $f \in L_q(\mathbb{R})$, $f \notin L_p(\mathbb{R})$

Theorem If $|\Omega| < \infty$, $0 < q < p$, $f \in L_p(\Omega)$ then

$$\|f\|_{L_q(\Omega)} \le |\Omega|^{1/q-1/p} \|f\|_{L_p(\Omega)}.$$

Proof Define $r := p/q \ge 1$

$$\|f\|_q^q = \int_\Omega |f|^q = \int_\Omega |f|^q 1 \underset{Holder}{\le} \left(\int_\Omega \left(|f|^q \right)^r \right)^{1/r} \left(\int_\Omega 1^{r'} \right)^{1/r'}$$

$$= \left(\int_\Omega |f|^p \right)^{q/p} |\Omega|^{1-q/p}$$

□

Thm Minkowski for Lp spaces $1 \le p \le \infty$, $f, g \in L_p$,

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \; .$$

Proof for $1 < p < \infty$ ($p = 1, \infty$ is esier)

$$\int (f+g)^p = \int f (f+g)^{p-1} + \int g (f+g)^{p-1}$$

$$\le \left(\|f\|_p + \|g\|_p \right) \left(\int (f+g)^{(p-1)p'} \right)^{1/p'}$$

$$= \left(\|f\|_p + \|g\|_p \right) \left(\int (f+g)^p \right)^{1-1/p}.$$

□

Thm For $0 < p < 1$, we have

(i) $$\left\| \sum_k f_k \right\|_p^p \le \sum_k \|f_k\|_p^p$$

(ii) $\|f+g\|_p \le 2^{1/p-1} \left(\|f\|_p + \|g\|_p \right)$ or in general $\left\| \sum_{k=1}^N f_k \right\|_p \le N^{1/p-1} \sum_{j=1}^N \|f_k\|_p$

Proof The quasi-norm (ii) is derived from (i), by using $1 \le p^{-1} < \infty$,

$$\left\|\sum_{k=1}^{N} f_k\right\|_p = \left(\sum_{j=1}^{N}\|f_k\|_p^p\right)^{1/p} = \left(\sum_{j=1}^{N}1\cdot\|f_k\|_p^p\right)^{1/p} \underset{Holder}{\leq} \left(\sum_{j=1}^{N}1^{\frac{1}{1-p}}\right)^{(1-p)1/p}\left(\sum_{j=1}^{N}\|f_k\|_p\right) = N^{1/p-1}\sum_{j=1}^{N}\|f_k\|_p$$

To prove (i), we need the following lemma

Lemma I For $0 < p \leq 1$ and any sequence of non-negative $a = \{a_k\}$,

$$\left(\sum_k a_k\right)^p \leq \sum_k a_k^p$$

Proof We first prove $(a_1 + a_2)^p \leq a_1^p + a_2^p$ and then apply induction.

To prove the inequality use $h(t) = t^p + 1 - (t+1)^p$. $h(0) = 0$ and $h'(t) = pt^{p-1} - p(t+1)^{p-1} \geq 0$. Therefore, $h(t) \geq 0$, for $t \geq 0$. This gives $t^p + 1 \geq (t+1)^p$. Setting $t = a_1 / a_2$ gives the desired inequality.

□

Proof of (i) : Simply apply the lemma pointwise for $x \in \mathbb{R}^n$

$$\left\|\sum_k f_k\right\|_p^p \leq \int_{\mathbb{R}^n}\left(\sum_k |f_k(x)|\right)^p dx \leq \int_{\mathbb{R}^n}\left(\sum_k |f_k(x)|^p\right)dx = \sum_k\int_{\mathbb{R}^n}|f_k(x)|^p dx = \sum_k \|f_k\|_p^p$$

□

Def The space $l_p(\mathbb{Z})$, $0 < p \leq \infty$, is the space of sequences $a = \{a_k\}_{k\in\mathbb{Z}}$, for which the norm is finite

$$\|a\|_{l_p} := \begin{cases}\left(\sum_k |a_k|^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_k |a_k|, & p = \infty.\end{cases}$$

Lemma II $l_p \subset l_q$ for $p \leq q$. That is, for any sequence $a = \{a_k\}$

$$\|a\|_{l_q} \leq \|a\|_{l_p} .$$

Proof Case of $q = \infty$, then

$$|a_j| = \left(|a_j|^p\right)^{1/p} \leq \left(\sum_k |a_k|^p\right)^{1/p} = \|a\|_{l_p} .$$

Therefore,

$$\|a\|_{l_\infty} = \sup_j |a_j| \leq \|a\|_{l_p} .$$

For $q < \infty$, we have

$$\left(\sum_k |a_k|^q\right)^{p/q} \leq \sum_k \left(|a_k|^q\right)^{p/q} = \sum_k |a_k|^p \Rightarrow \left(\sum_k |a_k|^q\right)^{1/q} \leq \left(\sum_k |a_k|^p\right)^{1/p} .$$

□