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On the analysis of anisotropic smoothness

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Abstract

We investigate anisotropic function spaces defined over the multi-level ellipsoid covers of $\mathbb{R}^n$, where the ellipsoids can quickly change shape from point to point and from level to level. We explicitly define an anisotropic modulus of smoothness (already used implicitly in Dahmen et al. (2010) [4]) and investigate its properties. We show anisotropic variants of classic inequalities such as the Marchaud, Nikolskii and Ul'yanov, relationships with isotropic smoothness and applications to anisotropic Besov space embedding.

Keywords: Anisotropic function spaces; Piecewise polynomial approximation

1. Introduction

Anisotropic function spaces on $\mathbb{R}^n$ were extensively studied, beginning with the Russian school in the 1960s (see Chapter 5 in [12] for a survey and references therein). In Section 2 we review a general anisotropic framework on $\mathbb{R}^n$ using the multi-level ellipsoid covers introduced in [4]. A discrete cover Θ, is a collection of multilevel covers $\{\Theta_m\}_{m\in Z}$, where each level is composed of ellipsoids of volume equivalent to 2^{-m}, with $\mathbb{R}^n = \bigcup_{\theta\in\Theta_m}\theta$. There are also specific conditions imposed on the change of the shape of the ellipsoids across space and scale (see (2.2)) as well as some other technical conditions. The ellipsoid covers induce anisotropic quasi-distances on $\mathbb{R}^n$ and together with the usual Lebesgue measure, form spaces of homogeneous type (see e.g. [8]).

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In Section 3 we review multi-level representations that are the building blocks required for anisotropic smoothness analysis. Section 4 contains an investigation of the properties of the anisotropic modulus of smoothness

$$\omega_{\Theta,r}\left(f,2^{-m}\right)_p := \begin{cases} \left(\sum_{\theta\in\Theta_m} \omega_r\left(f,\theta\right)_p^p\right)^{1/p}, & 0<p<\infty, \\ \sup_{\theta\in\Theta_m} \omega_r\left(f,\theta\right)_\infty, & p=\infty, \end{cases}$$

where $m \in \mathbb{Z}$ and $\omega_r(\cdot,\theta)_p := \omega_r(\cdot,\operatorname{diam}(\theta))_{L_p(\theta)}$, is the classic (isotropic) modulus of smoothness over the ellipsoid θ and $\operatorname{diam}(\Omega)$ is the diameter of a domain $\Omega \subset \mathbb{R}^n$. In the special case where the ellipsoids are equivalent to Euclidean balls, the above modulus is equivalent to the standard classic modulus over $L_p(\mathbb{R}^n)$ at the parameter $t = 2^{-m/n}$. We prove a Marchaud-type inverse inequality for $0 < p \le \infty$, $1 \le k < r$,

$$\omega_{\Theta,k}\left(f,2^{-m}\right)_p \le c2^{-a_6 mk}\left(\sum_{j=-\infty}^{m}\left[2^{a_6 jk}\omega_{\Theta,r}\left(f,2^{-j}\right)_p\right]^\gamma\right)^{1/\gamma},$$

where $\gamma := \min(1,p)$ and a_6 is one of the local 'anisotropy' parameters of the cover Θ (see (2.2)).

In Section 5, we prove an Ul'yanov-type inequality for $0 < p \le q \le \infty$,

$$\omega_{\Theta,r}\left(f,2^{-m}\right)_q \le c\left(\sum_{j=m}^{\infty} 2^{j\left(\frac{1}{p}-\frac{1}{q}\right)\gamma}\omega_{\Theta,r}\left(f,2^{-j}\right)_p^\gamma\right)^{1/\gamma},$$

where

$$\gamma := \begin{cases} q, & 0<q<\infty, \\ 1, & q=\infty. \end{cases}$$

In Section 6 we apply our Ul'yanov-type inequality to derive embedding results for the anisotropic Besov-type spaces [4].

Throughout the paper the constants $c > 0$ may vary from line to line and will depend on parameters such as the dimension n, the p norm, the parameters of the covers, but not on the functions or the scales m. We use the notation $\sim$ for equivalence between norms, metrics, or volumes of ellipsoids, etc. which means that each member of a pair can be bounded by an absolute constant times the other. For a measurable set $\Omega \subseteq \mathbb{R}^n$, the notation $|\Omega|$ implies the Lebesgue measure of Ω, and for a vector $v \in \mathbb{R}^n$, the notation $|v|$ implies the length of v. The norm of a matrix M, denoted by $\|M\|$, is the norm of M as an operator on l_2, i.e. $\sup_{|v|\le 1}|Mv|$. Finally, the notation $\#A$ means the number of elements in a finite set A.

2. Anisotropic ellipsoid covers of $\mathbb{R}^n$

We recall the definitions of Section 2 in [4]. The image of the Euclidean unit ball B^* in $\mathbb{R}^n$ via an affine transform will be called an ellipsoid. For a given ellipsoid θ we let A_θ be an affine transform such that $\theta = A_\theta(B^*)$. Denoting by $v_\theta := A_\theta(0)$ the 'center' of θ we have

$$A_\theta(x) = M_\theta x + v_\theta,$$

where M_θ is a nonsingular $n \times n$ matrix.

Definition 2.1. We say that

$$\Theta := \bigcup_{t\in\mathbb{R}} \Theta_t,$$

is a *continuous multilevel ellipsoid cover* of $\mathbb{R}^n$ if it satisfies the following conditions, where $p(\Theta) := \{a_1, \dots, a_6\}$ are positive constants:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ there exists an ellipsoid $\theta(x,t) \in \Theta_t$ and an affine transform $A_{x,t}(y) = M_{x,t}y + x$ such that $\theta(x,t) = A_{x,t}(B^*)$ and

$$a_1 2^{-t} \le |\theta(x,t)| \le a_2 2^{-t}.$$

(ii) For any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s > 0$, if $\theta(x,t) \cap \theta(y, t+s) \neq \emptyset$, then

$$a_3 2^{-a_4 s} \le 1/\left\| M_{y,t+s}^{-1} M_{x,t} \right\| \le \left\| M_{x,t}^{-1} M_{y,t+s} \right\| \le a_5 2^{-a_6 s}.$$

Definition 2.2. We call

$$\Theta = \bigcup_{m\in\mathbb{Z}} \Theta_m,$$

a *discrete multilevel ellipsoid cover* of $\mathbb{R}^n$ if the following conditions are obeyed, where $p(\Theta) := \{a_1, \dots, a_8, N_1\}$ are positive constants:

(a) Every level Θ_m, $m \in \mathbb{Z}$, consists of ellipsoids θ such that

$$a_1 2^{-m} \le |\theta| \le a_2 2^{-m}, \tag{2.1}$$

and Θ_m is a cover of $\mathbb{R}^n$, i.e. $\mathbb{R}^n = \bigcup_{\theta\in\Theta_m} \theta$.

(b) For each $\theta \in \Theta$ let A_θ be an affine transform associated with θ, of the form

$$A_\theta(x) = M_\theta x + v_\theta, \quad M_\theta \in \mathbb{R}^{n\times n},$$

such that $\theta = A_\theta(B^*)$ and $v_\theta = A(0)$ is the center of θ. We postulate that for any $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+\nu}$, $\nu \ge 0$, with $\theta \cap \theta' \neq \emptyset$, we have

$$a_3 2^{-a_4 \nu} \le 1/\left\| M_{\theta'}^{-1} M_\theta \right\| \le \left\| M_\theta^{-1} M_{\theta'} \right\| \le a_5 2^{-a_6 \nu}. \tag{2.2}$$

(c) Each $\theta \in \Theta_m$ can intersect with at most N_1 ellipsoids from Θ_m.

(d) For any $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, there exists $\theta \in \Theta_m$ such that $x \in \theta^\diamond$, where $\theta^\diamond$ is the dilated version of θ by a factor of $a_7 < 1$, i.e. $\theta^\diamond = A_\theta(a_7 B^*)$.

(e) If $\theta \cap \eta \neq \emptyset$ with $\theta \in \Theta_m$ and $\eta \in \Theta_m \cup \Theta_{m+1}$, then $\theta^\diamond \cap \eta^\diamond \neq \emptyset$, where $\theta^\diamond, \eta^\diamond$ are the dilated versions of θ, η by a factor a_7 as above.

The continuous and discrete ellipsoid covers induce quasi-distances on R^n. A *quasi-distance* on a set X is a mapping $\rho : X \times X \to [0,\infty)$ that satisfies the following conditions:

(i) $\rho(x,y) = 0 \Leftrightarrow x = y$,

(ii) $\rho(x,y) = \rho(y,x)$,

(iii) There exists $\kappa \ge 1$, such that for all $x, y, z \in \mathbb{R}^n$,

$$\rho(x,y) \le \kappa(\rho(x,z) + \rho(z,y)). \tag{2.3}$$

Let Θ be a cover. We define $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$\rho(x, y) = \inf_{\theta \in \Theta} \{|\theta| : x, y \in \theta\}. \tag{2.4}$$

The following results are proved in [4].

Theorem 2.3. *The function ρ in (2.4), induced by a discrete or a continuous ellipsoid cover, is a quasi-distance on $\mathbb{R}^n$.*

Let Θ be an ellipsoid cover inducing a quasi-distance ρ. We denote $B(x, r) := \cup\{y \in \mathbb{R}^n : \rho(x, y) < r\}$. Evidently,

$$B(x, r) = \bigcup_{\theta \in \Theta} \{\theta : |\theta| \le r,\ x \in \theta\}.$$

Theorem 2.4. *Let Θ be an ellipsoid cover. For each ball $B(x, r)$, there exist ellipsoids $\theta', \theta'' \in \Theta$, such that $\theta' \subset B(x, r) \subset \theta''$ and $|\theta'| \sim |B(x, r)| \sim |\theta''| \sim r$, where the constants depend on $p(\Theta)$.*

In this paper, our definitions and results are given for discrete covers. However, this is justified by the following

Theorem 2.5. *For every continuous cover Θ there exists a discrete cover $\hat{\Theta}$ such that*

(i) *$\hat{\Theta}$ satisfies all of the requirements of Definition 2.2, with parameters that depend on the parameters of Θ.*
(ii) *Any ellipsoid $\hat{\theta} \in \hat{\Theta}$, is obtained from a certain ellipsoid in Θ through a dilation by a fixed factor.*
(iii) *The induced quasi-distances are equivalent, i.e. for any $x, y \in \mathbb{R}^n$, $\rho(x, y) \sim \hat{\rho}(x, y)$.*

Spaces of homogeneous type were first introduced in [3] (see [8] for a survey of harmonic analysis on spaces of homogeneous type) as a means to extend the Calderon–Zygmund theory of singular integral operators to more general settings. Let X be a topological space endowed with a Borel measure μ and a quasi-distance ρ. Assume that the balls $B(x, r) := \{y \in X : \rho(x, y) < r\}, x \in X, r > 0$, form a basis for the topology in X. Then, the space (X, ρ, μ) is said to be of *homogeneous type* if there exists a constant A such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \le A\mu(B(x, r)). \tag{2.5}$$

If (2.5) holds then μ is said to be a *doubling measure*. A space of homogeneous type is said to be *normal*, if the equivalence $\mu(B(x, r)) \sim r$ holds. Indeed, Theorem 2.4 implies that an ellipsoid cover induces a normal space of homogeneous type $(\mathbb{R}^n, \rho, dx)$, where ρ is the quasi-distance (2.4) and dx is the Lebesgue measure.

Let us describe a useful form of covers of $\mathbb{R}^2$. We select all ellipses on levels ≤ 0 to be Euclidean balls, which is a special case of what we define below a 'quasi zero-uniform cover' (see Definition 4.7). For levels >0 we allow the ellipses to change from Euclidean balls to ellipses with the 'parabolic scaling' parameters $(a_6, a_4) = (1/3, 2/3)$. This choice of these parameters relates to polygonal approximation of a planar curve, with segments of length h and approximation error of $O(h^2)$. Roughly speaking, with this choice we can simulate the performance of polygonal approximation by constructing at the level $m > 0$ roughly $O(2^{m/3})$

'thin' ellipses of length $\sim 2^{-m/3}$ and width $\sim 2^{-2m/3}$, such they (are aligned with and) cover the function's curve singularities with a 'strip width' of $\sim 2^{-2m/3}$. The actual number of ellipses that are needed depends on the total length of the curve singularities as well as their 'curve smoothness'. Away from the curve singularities, the ellipses can be selected to be Euclidean balls (see also the constructions in Section 7.1 of [4]).

We conclude this section by relating the quasi-distances induced by ellipsoid covers with the Euclidean distance. To this end we first require the following definition.

Definition 2.6. Let ρ be a quasi-distance on $\mathbb{R}^n$ and let $\mu = (\mu_0, \mu_1)$, $0 < \mu_0 \le \mu_1 < \infty$. For any $x, y \in \mathbb{R}^n$ and $d > 0$ we define

$$\mu(x, y, d) := \begin{cases} \mu_0 & \rho(x, y) < d, \\ \mu_1 & \rho(x, y) \ge d. \end{cases} \qquad \tilde{\mu}(x, y, d) := \begin{cases} \mu_1 & \rho(x, y) < d, \\ \mu_0 & \rho(x, y) \ge d. \end{cases} \tag{2.6}$$

Theorem 2.7 (*[5]*). *Let Θ be a discrete ellipsoid cover and ρ the quasi-distance (2.4). Denote by $\mu := (\mu_0, \mu_1) = (a_6, a_4)$ where $0 < a_6 \le a_4$ are the parameters from Definition 2.2. Then for each fixed $y \in \mathbb{R}^n$ there exist constants $0 < c_1 < c_2 < \infty$ that depend on y and $p(\Theta)$ such that*

$$c_1 \rho(x, y)^{\tilde{\mu}(x,y,1)} \le |x - y| \le c_2 \rho(x, y)^{\mu(x,y,1)}, \quad \forall x \in \mathbb{R}^n, \tag{2.7}$$

where $|x - y|$ is the usual Euclidean distance between x and y.

In the special case where the ellipsoid cover is composed of Euclidean balls, we have that the parameters in (2.2) satisfy $a_4 = a_6 = 1/n$ and (2.7) is easily verified by

$$\begin{aligned} |x - y| &\sim |\{z : |z - x| \le |y - x|\}|^{1/n} \sim \rho(x, y)^{1/n} \\ &= \rho(x, y)^{\mu(x,y,1)} = \rho(x, y)^{\tilde{\mu}(x,y,1)}. \end{aligned}$$

3. Anisotropic multiresolution analysis

3.1. Polynomials on convex domains

Denote by Π_{r-1} the polynomials of total degree $r - 1$. We will require the following facts concerning polynomials over convex domains. The first two are Lemmas 3.1 and 3.2 from [7]

Proposition 3.1. *Let $P \in \Pi_{r-1}$ and let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ be bounded convex domains such that $|\Omega_2| \le \rho |\Omega_1|$ for some $\rho > 1$. Then, for $0 < p \le \infty$,*

$$\|P\|_{L_p(\Omega_2)} \le c(n, r, p, \rho) \|P\|_{L_p(\Omega_1)}. \tag{3.1}$$

Proposition 3.2. *Let $P \in \Pi_{r-1}$ and let $\Omega \subset \mathbb{R}^n$ be bounded convex domain. Then, for any $0 < p, q \le \infty$,*

$$\|P\|_{L_q(\Omega)} \sim |\Omega|^{1/q-1/p} \|P\|_{L_p(\Omega)}, \tag{3.2}$$

with constants that only depend on n, r, p and q.

The third required result is the Markov-type inequality [11]

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for $1 \le p \le \infty$, any polynomial $P \in \Pi_{r-1}$, and $\beta \in \mathbb{Z}_+^n$, $|\beta| := \sum_{i=1}^n \beta_i \le r-1$,*

$$\left\|\partial^\beta P\right\|_{L_p(\Omega)} \le C(n,|\beta|)\,\mathrm{width}(\Omega)^{-|\beta|}\,\|P\|_{L_p(\Omega)}, \tag{3.3}$$

where width (Ω) *is the diameter of the largest n-dimensional Euclidean ball that is contained in Ω.*

3.2. Compactly supported, polynomial reproducing 'bumps'

Let Θ be a discrete ellipsoid cover (see Definition 2.2), possibly sampled from a continuous cover (see Theorem 2.5). We shall first construct for each level $m \in \mathbb{Z}$ a stable basis Φ_m whose elements are C^∞ 'bumps' that reproduce polynomials and are supported on the ellipsoids of Θ_m. To this end, we split Θ_m into no more than N_1 disjoint sets $\left\{\Theta_m^\nu\right\}_{\nu=1}^{N_1}$ (N_1 appears in condition (c) in Definition 2.2), so that neither two ellipsoids $\theta', \theta'' \in \Theta_m$, with $\theta' \cap \theta'' \ne \emptyset$ are of the same color.

Remark 3.4. In Section 3.4, where we require the stability of the 'two-level splits' of [4], we shall need a stronger coloring scheme, where two intersecting ellipsoids from adjacent levels also have different colors.

For a fixed order $r \in \mathrm{N}$, there exist functions $\phi_\nu \in C^\infty(\mathbb{R}^n)$, $1 \le \nu \le N_1$, with the following properties (Section 3 of [5]):

(i) $\phi_\nu \ge 0$ with $\operatorname{supp}\phi_\nu = \overline{B^*}$ where B^* is the Euclidean unit ball in $\mathbb{R}^n$.
(ii) The restriction of ϕ_ν is a polynomial of degree $2\nu r$ on $(a_7+1)/2B^*$.
(iii) In addition

$$\phi_\nu|_{a_7 B^*} \ge c_1 > 0, \quad c_1 = c_1(N_1, r). \tag{3.4}$$

For any ellipsoid θ let A_θ be an affine transform satisfying $A_\theta(B^*) = \theta$. Here, we use our ellipsoid coloring scheme to define $\phi_\theta := \phi_\nu \circ A_\theta^{-1}$, if $\theta \in \Theta_m^\nu$. It is standard to form a partition of unity $\left\{\tilde{\phi}_\theta\right\}_{\theta\in\Theta_m}$ by setting

$$\tilde{\phi}_\theta := \frac{\phi_\theta}{\sum_{\theta'\in\Theta_m} \phi_{\theta'}}. \tag{3.5}$$

Observe that property (d) of ellipsoid covers (see Definition 2.2) together with (3.4) ensure the existence of constants $0 < c' \le c'' < \infty$, such that $c' \le \sum_{\theta\in\Theta_m} \varphi_\theta(x) \le c''$, for all $x \in \mathbb{R}^n$ and hence $\left\{\tilde{\phi}_\theta\right\}$ are well defined and satisfy the partition of unity

$$\sum_{\theta\in\Theta_m} \tilde{\phi}_\theta = 1. \tag{3.6}$$

By property (ii) above, the 'core' part of each $\tilde{\phi}_\theta$ is a rational function, whose nominator is a polynomial of a certain degree which is different from the degrees of the nominators of its neighbors, i.e. the basis functions supported on neighbor ellipsoids. This construction gives local linear independence of neighbor basis function and eventually leads to the global stability of Φ_m.

Fix $1 \le \nu \le N_1$ and suppose

$$\left\{P_\beta : \beta \in \mathbb{N}^n, |\beta| = \beta_1 + \cdots + \beta_n \le r-1\right\},$$

is an orthonormal basis for Π_{r-1} in the weighted norm $\|f\|_{L_2(B^*,\phi_\nu)} := \|f\phi_\nu\|_{L_2(B^*)}$. Then for any $\theta \in \Theta_m^\nu$ and $\beta \in \mathbb{N}^n, |\beta| < r$, we define

$$P_{\theta,\beta} := |\theta|^{-1/2} P_\beta \circ A_\theta^{-1}, \tag{3.7}$$

and set

$$\varphi_{\theta,\beta} := P_{\theta,\beta}\tilde{\phi}_\theta. \tag{3.8}$$

To simplify our notation, we denote

$$\Lambda_m := \{\lambda := (\theta, \beta) : \theta \in \Theta_m, |\beta| < r\}, \tag{3.9}$$

and if $\lambda = (\theta, \beta)$ we shall denote by θ_λ and β_λ the components of λ.

Notice that from our construction $\|\varphi_\lambda\|_2 = 1$ and in general $\|\varphi_\lambda\|_p \sim |\theta_\lambda|^{1/p-1/2}, 0 \le p \le \infty$. In going further we define the mth level basis Φ_m by $\Phi_m := \{\varphi_\lambda : \lambda \in \Lambda_m\}$ and set $\mathcal{S}_m^r := \overline{\operatorname{span}}(\Phi_m)$. It is easy to see that $\Pi_{r-1} \subset \mathcal{S}_m^r$, since for any polynomial $P \in \Pi_{r-1}$ and $\theta \in \Theta_m$ there exist a representation $P = \sum_{|\beta|<r} c_{\theta,\beta} P_{\theta,\beta}$ and therefore, by the partition of unity (3.6)

$$P = \sum_{\theta\in\Theta_m} P\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m,|\beta|<r} c_{\theta,\beta} P_{\theta,\beta}\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m,|\beta|<r} c_{\theta,\beta}\varphi_{\theta,\beta} = \sum_{\lambda\in\Lambda_m} c_\lambda\varphi_\lambda. \tag{3.10}$$

As we already discussed, the stability of Φ_m is critical for our further development.

Theorem 3.5 (*Proposition 3.1 in [5]*). *For $f \in \mathcal{S}_m^r \cap L_p, 0 < p \le \infty$, with $f = \sum_{\lambda\in\Lambda_m} c_\lambda\varphi_\lambda$, the following holds*

$$\|f\|_p \sim \begin{cases} \left(\sum_{\lambda\in\Lambda_m} \|c_\lambda\varphi_\lambda\|_p^p\right)^{1/p} \sim 2^{m\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\sum_{\lambda\in\Lambda_m} |c_\lambda|^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_{\lambda\in\Lambda_m} \|c_\lambda\varphi_\lambda\|_\infty \sim 2^{m/2} \sup_{\lambda\in\Lambda_m} |c_\lambda|, & p = \infty, \end{cases} \tag{3.11}$$

where the constants of equivalency depend only on $p(\Theta), n, r, p$ and our choice of 'bumps' $\{\phi_\nu\}_{\nu=1,\ldots,N_1}$.

3.3. Compactly supported duals and projectors

Let $\Omega \subseteq \mathbb{R}^n$ be a subdomain with non empty interior. For $f \in L_p(\Omega), 0 < p \le \infty, h \in \mathbb{R}^d$ and $r \in \mathbb{N}$ we recall the rth order difference operator $\Delta_h^r(f) : \Omega \to \mathbb{R}$

$$\Delta_h^r(f,x) := \Delta_h^r(f,x,\Omega) := \begin{cases} \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x+kh) & [x, x+rh] \subset \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

where $[x, y]$ denotes the line segment connecting any two points $x, y \in \mathbb{R}^n$. The *modulus of smoothness of order r* of a function in $L_p(\mathbb{R}^n)$ is defined by

$$\omega_r(f,t)_p := \sup_{|h|\le t} \left\|\Delta_h^r(f,\cdot)\right\|_{L_p(\mathbb{R}^n)}, \quad t > 0. \tag{3.12}$$

For $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < p \le \infty$, and any bounded convex domain $\Omega \subset \mathbb{R}^n$ we denote

$$\omega_r(f, \Omega)_p := \omega_r(f, \operatorname{diam}(\Omega))_{L_p(\Omega)}. \tag{3.13}$$

Next, for any $\theta \in \Theta$, let $T_{\theta,p} : L_p(\theta) \to \Pi_{r-1}$ be a projector such that

$$\left\| f - T_{\theta,p} f \right\|_{L_p(\theta)} \le c(n,r,p)\, \omega_r(f,\theta)_p, \quad f \in L_p(\theta). \tag{3.14}$$

By the Whitney theorem (see [7] for results on arbitrary convex domains), $T_{\theta,p}$ can be defined as the best or a near best approximation to f from Π_{r-1} in $L_p(\theta)$. For $p \ge 1$ the local projectors $T_{\theta,p}$ can be realized as a linear operator using the Averaged Taylor polynomials (see e.g. [6]), but for $0 < p < 1$, $T_{\theta,p}$ are not linear operators. Forming a partition of unity of these local polynomial approximations on each level gives the following operators

$$T_m(f) := T_{m,p}(f) := \sum_{\theta \in \Theta_m} T_{\theta,p}(f)\, \tilde{\phi}_\theta, \quad m \in \mathbb{Z}. \tag{3.15}$$

Lemma 3.6 (*Lemma 3.4 [4]*). *Let Θ be a discrete cover. Then for any $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < p \le \infty$,*

(i) $\|T_m f\|_{L_p(\theta)} \le c \|f\|_{L_p(\theta^*)}$, *for any $\theta \in \Theta_m$, where $\theta^* := \bigcup_{\theta' \in \Theta_m, \theta \cap \theta' \ne \emptyset} \theta'$.*
(ii) $\|f - T_m f\|_{L_p(\theta)} \le c \sum_{\theta' \in \Theta_m : \theta' \cap \theta \ne \emptyset} \omega_r(f, \theta')_p$.
(iii) $\|f - T_m f\|_{L_p(\Omega)} \to 0$, *as $m \to \infty$, for any compact $\Omega \subset \mathbb{R}^n$.*

3.4. Two level splits

In this section we recall the 'two level split' system from [4]. Denote

$$\mathcal{M}_m := \{\nu = (\eta, \theta, \beta) : \eta \in \Theta_{m+1}, \theta \in \Theta_m, \eta \cap \theta \ne \emptyset, |\beta| < r\}, \quad m \in \mathbb{Z},$$

and define using (3.5) and (3.8)

$$F_\nu := P_{\eta,\beta} \tilde{\phi}_\eta \tilde{\phi}_\theta = \varphi_{\eta,\beta} \tilde{\phi}_\theta, \quad \nu \in \mathcal{M}_m. \tag{3.16}$$

We also denote $\mathcal{F}_m := \{F_\nu : \nu \in \mathcal{M}_m\}$ and set $W_m := \operatorname{span}(\mathcal{F}_m)$. Note that $F_\nu \in C^\infty$, $\operatorname{supp}(F_\nu) = \theta \cap \eta$ if $\nu = (\eta, \theta, \beta)$, and by property (e) in Definition 2.2 we have that $\|F_\nu\|_p \sim |\eta|^{1/p - 1/2}$, $0 < p \le \infty$. It is important that with careful construction (see Remark 3.4) $\mathcal{F}_m$ is a also a stable basis.

Theorem 3.7 (*Theorem 4.2 [4]*). *If $f \in W_m \cap L_p(\mathbb{R}^n)$, $0 < p \le \infty$, and $f = \sum_{\nu \in \mathcal{M}_m} a_\nu F_\nu$, then*

$$\|f\|_p \sim \begin{cases} \left(\sum_{\nu \in \mathcal{M}_m} \|a_\nu F_\nu\|_p^p \right)^{1/p} \sim 2^{m\left(\frac{1}{2} - \frac{1}{p}\right)} \left(\sum_{\nu \in \mathcal{M}_m} |a_\nu|^p \right)^{1/p}, & 0 < p < \infty, \\ \sup_{\nu \in \mathcal{M}_m} \|a_\nu F_\nu\|_\infty \sim 2^{m/2} \sup_{\nu \in \mathcal{M}_m} |a_\nu|, & p = \infty. \end{cases} \tag{3.17}$$

Let the coefficients $\left\{ A_{\alpha,\beta}^{\theta,\eta} \right\}$ be determined from

$$P_{\theta,\alpha} = \sum_{|\beta| < r} A_{\alpha,\beta}^{\theta,\eta} P_{\eta,\beta}, \quad \theta \in \Theta_m,\ \eta \in \Theta_{m+1}. \tag{3.18}$$

For any $\lambda = (\theta, \alpha) \in \Lambda_m$ we obtain, using (3.18) and (3.16), the following *meshless two-scale relationship*

$$\begin{aligned} \phi_\lambda &= P_{\theta,\alpha}\tilde{\varphi}_\theta = \sum_{\eta\in\Theta_{m+1},\eta\cap\theta\neq\emptyset} P_{\theta,\alpha}\tilde{\varphi}_\theta\tilde{\varphi}_\eta \\ &= \sum_{\eta\in\Theta_{m+1},\eta\cap\theta\neq\emptyset,|\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}\tilde{\varphi}_\theta\tilde{\varphi}_\eta = \sum_{\eta\in\Theta_{m+1},\eta\cap\theta\neq\emptyset,|\beta|<r} A^{\theta,\eta}_{\alpha,\beta} F_{\eta,\theta,\beta}, \end{aligned}$$

and hence $\varphi_\lambda \in W_m$. Also, if $\lambda \in \Lambda_{m+1}$ and $\lambda = (\eta, \beta)$, then

$$\varphi_\lambda = P_{\eta,\beta}\tilde{\phi}_\eta = \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} P_{\eta,\beta}\tilde{\phi}_\eta\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} F_{\eta,\theta,\beta}.$$

Combining the last two results we find that $\overline{\text{span}}\,(\Phi_m \cup \Phi_{m+1}) \subset W_m$.

4. An anisotropic modulus of smoothness

4.1. Properties

Definition 4.1. Let Θ be a discrete cover. For any $r \in \mathbb{N}$ and $m \in \mathbb{Z}$, we define the *anisotropic modulus of smoothness* by

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_p := \begin{cases} \left(\sum_{\theta\in\Theta_m} \omega_r\,(f,\theta)_p^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_{\theta\in\Theta_m} \omega_r\,(f,\theta)_\infty\,, & p = \infty, \end{cases} \tag{4.1}$$

where $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and $\omega_r(\cdot,\theta)_p$ is defined in (3.13).

Although the underlying geometry can possibly be highly anisotropic, the modulus of (4.1) has similar properties to the classic isotropic modulus (3.12).

Theorem 4.2. *Let Θ be a cover inducing a quasi-distance ρ. The modulus $\omega_{\Theta,r}(\cdot,\cdot)_p$ has the following properties:*

(a) *There exists a constant $c(r, N_1)$ such that for any $f \in L_p(\mathbb{R}^n)$, $0 < p \le \infty$, $\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \le c\,\|f\|_p$, $\forall m \in \mathbb{Z}$.*

(b) *For any $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$, we have that $\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \to 0$ as $m \to \infty$.*

(c) *For $r \in \mathbb{N}$ $0 < p \le \infty$ there exists a constant $\lambda(\Theta, r, p) \ge 1$ such that for any $f \in L_p(\mathbb{R}^n)$, $m \in \mathbb{Z}$ and $k \ge 1$*

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \le \lambda^k \omega_{\Theta,r}\left(f, 2^{-(m+k)}\right)_p. \tag{4.2}$$

(d) *If another discrete cover $\tilde{\Theta}$ induces an equivalent quasi-distance $\tilde{\rho}$, i.e. $c_1\rho(x,y) \le \tilde{\rho}(x,y) \le c_2\rho(x,y)$, for all $x, y \in \mathbb{R}^n$, then for any $r \in \mathbb{N}$, $0 < p \le \infty$, and $m \in \mathbb{Z}$,*

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \sim \omega_{\tilde{\Theta},r}\left(f, 2^{-m}\right)_p, \tag{4.3}$$

where the constants of equivalency depend only on c_1, c_2 and the parameters of the covers.

Proof. (a) The fact that $\omega_{\Theta,r}(f,\cdot)_p$ is bounded by $c(r,N_1)\|f\|_p$ is obvious from the fact that each ellipsoid $\theta\in\Theta_m$ intersects with at most N_1 neighbors from Θ_m and the known bound $\omega_r(f,\theta)_p\le c\|f\|_{L_p(\theta)}$.
(b) For any $\varepsilon>0$ let $Q_\varepsilon:=[-M,M]^n, M>0$, such that $\int_{\mathbb{R}^n\setminus Q_\varepsilon}|f|^p\,dx\le\varepsilon$. Let $d_0:=\max_{\theta\in\Theta_0,\theta\cap Q_\varepsilon\ne\emptyset}\operatorname{diam}(\theta)$. From (2.2) we get for any $\theta\in\Theta_m, m\ge 0$, that if $\theta\cap Q_\varepsilon\ne\emptyset$, then $\operatorname{diam}(\theta)\le cd_0 2^{-a_4 m}$. This 'quasi-uniform' property on the compact set, Q_ε, ensures that, as in the uniform (isotropic) case

$$\sum_{\theta\in\Theta_m,\theta\cap Q_\varepsilon\ne\emptyset}\omega_r(f,\theta)_p^p\to 0,\quad \text{as } m\to\infty.$$

We also have

$$\sum_{\theta\in\Theta_m,\theta\cap Q_\varepsilon=\emptyset}\omega_r(f,\theta)_p^p\le c\sum_{\theta\in\Theta_m,\theta\cap Q_\varepsilon=\emptyset}\|f\|_{L_p(\theta)}^p\le c\|f\|_{L_p(\mathbb{R}^n\setminus Q_\varepsilon)}^p\le c\varepsilon.$$

(c) It is sufficient to prove that $\omega_{\Theta,r}\left(f,2^{-m}\right)_p\le\lambda\omega_{\Theta,r}\left(f,2^{-(m+1)}\right)_p$, since the general case (4.2) follows by repeated application. Also, as the properties from Definition 2.2 imply that for any $\theta\in\Theta_m$,

$$\#\{\eta\in\Theta_{m+1}:\eta\cap\theta\ne\emptyset\}\le L,$$

it is sufficient to show that there exists a constant $\tilde\lambda=\tilde\lambda(\Theta,r,p)$, such that for each $\theta\in\Theta_m$

$$\omega_r(f,\theta)_p\le\tilde\lambda\begin{cases}\left(\displaystyle\sum_{\eta\in\Theta_{m+1}:\theta\cap\eta\ne\emptyset}\omega_r(f,\eta)_p^p\right)^{1/p}, & 0<p<\infty,\\ \displaystyle\max_{\eta\in\Theta_{m+1}:\theta\cap\eta\ne\emptyset}\omega_r(f,\eta)_\infty, & p=\infty.\end{cases}\tag{4.4}$$

Assume first that $m=0$ and $\theta=B^*$ (the Euclidean unit ball). From (2.2) it follows that each $\eta\in\Theta_1,\eta\cap\theta\ne\emptyset$ is 'equivalent' to a Euclidean ball with constants that depend on $p(\Theta)$. Moreover, property (d) in Definition 2.2 implies that there exists a constant $\tilde c(\Theta)$ such that for each $x\in\theta$, there exists $\eta\in\Theta_1$ such that $\{y\in\mathbb{R}^n:|x-y|\le\tilde c\}\subseteq\eta$. From (3.13), $\omega_r(f,\theta)_p=\omega_r(f,B^*)_p=\sup_{|h|\le 2/r}\left\|\Delta_h^r(f,\cdot)\right\|_{L_p(B^*)}$. Observe that for any $h\in\mathbb{R}^n$, $|h|\le 2/r$, there exists an integer $J\le 2\left\lceil\tilde c^{-1}\right\rceil$ and $\tilde h\in\mathbb{R}^n$, $\left|\tilde h\right|\le\tilde c/r$, such that $h=J\tilde h$. Using a well-known identity for the difference operator (see e.g. Chapter 2 in [9]), we have

$$\Delta_h^r(f,x)=\sum_{k_1=0}^{J-1}\cdots\sum_{k_r=0}^{J-1}\Delta_{\tilde h}^r\left(f,x+k_1\tilde h+\cdots+k_r\tilde h\right).$$

For any domain $\Omega\subseteq\mathbb{R}^n$, denote $X(\Omega,h):=\{x\in\Omega:[x,x+rh]\subset\Omega\}$. Then, if $x\in X(B^*,h)$, then also $x+k_1\tilde h+\cdots+k_r\tilde h\in B^*$, for any $0\le k_1,\ldots,k_r<J$. Furthermore, since $r|\tilde h|\le\tilde c$, for any $y\in B^*$, there exists some $\eta\in\Theta_1,\eta\cap B^*\ne\emptyset$, such that $[y,y+r\tilde h]\subset\eta$. From this we conclude that for $0<p<\infty$ and any $h\in\mathbb{R}^n$, $|h|\le 2/r$, there exists a constant $\tilde\lambda>0$ such that

$$\int_{B^*}\left|\Delta_h^r\left(f,x,B^*\right)\right|^p dx=\int_{X(B^*,h)}\left|\Delta_h^r(f,x)\right|^p dx$$

$$\leq \tilde{\lambda}^p \sum_{\eta \in \Theta_1 : \eta \cap B^* \neq \emptyset} \int_{X\left(\eta, \tilde{h}\right)} \left| \Delta_{\tilde{h}}^r (f, x) \right|^p dx$$

$$\leq \tilde{\lambda}^p \sum_{\eta \in \Theta_1 : \eta \cap B^* \neq \emptyset} \omega_r (f, \eta)_p^p .$$

This proves (4.4) for the case $m = 0, \theta = B^*$ and $0 < p < \infty$ (the case $p = \infty$ is similar). In the case where Θ is an arbitrary cover and $\theta \in \Theta_m$, let $\tilde{\Theta} := A_\theta^{-1} (\Theta)$, where $A_\theta (x) = Mx + v$ is an affine transform satisfying $A_\theta (B^*) = \theta$. Observe that $\tilde{\Theta}$ is a discrete cover with parameters equivalent to $p(\Theta)$. Denoting $\tilde{f} := f(A_\theta \cdot)$ we have

$$\begin{aligned}
\omega_r (f, \theta)_p^p &= |\det (M)| \, \omega_r \left(\tilde{f}, B^*\right)_p^p \\
&\leq \tilde{\lambda}^p |\det (M)| \sum_{\tilde{\eta} \in \tilde{\Theta}_1 : B^* \cap \tilde{\eta} \neq \emptyset} \omega_r \left(\tilde{f}, \tilde{\eta}\right)_p^p \\
&\leq \tilde{\lambda}^p \sum_{\eta \in \Theta_{m+1} : \theta \cap \eta \neq \emptyset} \omega_r (f, \eta)_p^p .
\end{aligned}$$

This proves (4.4) and completes the proof of (4.2) for $1 \leq p < \infty$. The proof for $p = \infty$ is similar.

(d) Let $\Theta, \tilde{\Theta}$ be two discrete covers with parameters $p(\Theta), p(\tilde{\Theta})$ and equivalent induced quasi-distances $\rho \sim \tilde{\rho}$. We claim that there exists a constant $J \geq 1$ that depends on the parameters of the covers, such that for each $\theta \in \Theta_m$ there exists $\tilde{\theta} \in \tilde{\Theta}_{m-J}$, such that $\theta \subset \tilde{\theta}$. Indeed, by the properties of discrete covers, $\theta \subseteq B\left(v_\theta, a_2 2^{-m}\right)$. The equivalence of the quasi-distances implies the existence of a uniform constant $\tilde{c} > 0$, for which $B\left(v_\theta, a_2 2^{-m}\right) \subseteq \tilde{B}\left(v_\theta, \tilde{c} a_2 2^{-m}\right)$, where $\tilde{B}$ is an anisotropic Ball corresponding to the quasi-distance induced by $\tilde{\Theta}$. Next, by Theorem 2.4, there exist $\tilde{\theta} \in \tilde{\Theta}$ such that $\tilde{B}\left(v_\theta, \tilde{c} a_2 2^{-m}\right) \subseteq \tilde{\theta}$ and $|\tilde{\theta}| \sim \tilde{c} a_2 2^{-m}$. This gives $\theta \subseteq \tilde{\theta}$, with $|\theta| \sim |\tilde{\theta}|$. Evidently for any $f \in L_p^{\mathrm{loc}}$, $\omega_r (f, \theta)_p \leq \omega_r (f, \tilde{\theta})_p$. Applying (4.2) we conclude that

$$\omega_{\Theta,r} \left(f, 2^{-m}\right)_p \leq c \omega_{\tilde{\Theta},r} \left(f, 2^{-m+J}\right)_p \leq c \omega_{\tilde{\Theta},r} \left(f, 2^{-m}\right)_p . \quad \square$$

With the anisotropic modulus defined, we are able to state that a direct consequence of Lemma 3.6 and the finite intersection property (c) of Definition 2.2, is the following Jackson-type result.

Theorem 4.3. *For a cover* Θ, $1 \leq k \leq r$, $0 < p \leq \infty$, *and any* $m \in \mathbb{Z}$,

$$\|f - T_m f\|_p \leq c \omega_{\Theta,k} \left(f, 2^{-m}\right)_p . \tag{4.5}$$

Our last result for this subsection is the Marchaud inequality for the anisotropic modulus (see Theorem 8.1, Chapter 2 in [9] for the isotropic case).

Theorem 4.4. *For a cover* Θ, $1 \leq k < r$ *and* $0 < p \leq \infty$, *the following holds for any* $m \in \mathbb{Z}$,

$$\omega_{\Theta,k} \left(f, 2^{-m}\right)_p \leq c 2^{-a_6 m k} \left(\sum_{j=-\infty}^{m} \left[2^{a_6 j k} \omega_{\Theta,r} \left(f, 2^{-j}\right)_p \right]^\gamma \right)^{1/\gamma} , \tag{4.6}$$

where $\gamma := \min(1, p)$ *and* a_6 *is defined in* (2.2).

Proof. Assume first that $0 < p < \infty$. We use a telescopic sum of the operators $\{T_j\}$ from Section 3.3 which provide 'local' approximation order r and apply Theorem 4.2(a) and (4.5) to obtain

$$\omega_{\Theta,k}\left(f,2^{-m}\right)_p^\gamma \le \omega_{\Theta,k}\left(f-T_mf,2^{-m}\right)_p^\gamma + \sum_{j=-\infty}^{m}\omega_{\Theta,k}\left(T_jf-T_{j-1}f,2^{-m}\right)_p^\gamma$$
$$\le c\left(\omega_{\Theta,r}\left(f,2^{-m}\right)_p^\gamma + \sum_{j=-\infty}^{m}\omega_{\Theta,k}\left(\left(T_j-T_{j-1}\right)f,2^{-m}\right)_p^\gamma\right).$$

It remains to show that

$$\omega_{\Theta,k}\left(\left(T_j-T_{j-1}\right)f,2^{-m}\right)_p \le c2^{a_6k(j-m)}\omega_{\Theta,r}\left(f,2^{-j}\right)_p, \quad j\le m. \tag{4.7}$$

Assume $\left(T_j-T_{j-1}\right)f=\sum_{\nu\in\mathcal{M}_j}c_\nu F_\nu$ is the 'two-level split' representation of Section 3.4. By (A.4) in [4], for any $\theta\in\Theta_m$ and $F_\nu\in\mathcal{F}_j$, $j\le m$, such that $\theta\cap\eta_\nu\neq\emptyset$

$$\omega_k\left(F_\nu,\theta\right)_p^p \le c\,|\theta|\,|\eta_\nu|^{-1}\,2^{-a_6k(m-j)p}\,\|F_\nu\|_p^p \le c2^{j-m-a_6k(m-j)p}\,\|F_\nu\|_p^p.$$

From this, the properties of covers, Theorem 3.5 and (4.5), we conclude (4.7)

$$\omega_{\Theta,k}\left(\left(T_j-T_{j-1}\right)f,2^{-m}\right)_p^p = \sum_{\theta\in\Theta_m}\omega_k\left(\left(T_j-T_{j-1}\right)f,\theta\right)_p^p$$
$$\le c\sum_{\theta\in\Theta_m}\left(\sum_{\nu\in\mathcal{M}_j:\theta\cap\eta_\nu\neq\emptyset}\omega_k\left(c_\nu F_\nu,\theta\right)_p\right)^p$$
$$\le c2^{j-m-a_6k(m-j)p}\sum_{\theta\in\Theta_m}\left(\sum_{\nu\in\mathcal{M}_j:\theta\cap\eta_\nu\neq\emptyset}\|c_\nu F_\nu\|_p\right)^p$$
$$\le c2^{j-m-a_6k(m-j)p}\left(\max_{\eta\in\Theta_j}\#\left\{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset\right\}\right)$$
$$\times\sum_{\nu\in\mathcal{M}_j}\|c_\nu F_\nu\|_p^p$$
$$\le c2^{-a_6k(m-j)p}\left\|\left(T_j-T_{j-1}\right)f\right\|_p^p$$
$$\le c2^{-a_6k(m-j)p}\omega_{\Theta,r}\left(f,2^{-j}\right)_p^p.$$

The proof for the case $p=\infty$ is similar. □

4.2. Relationship with isotropic smoothness

For the next result we need the following

Proposition 4.5 (*Theorem 7.1 in [10]*). *Suppose the following conditions hold for a convex domain $\Omega\subseteq\mathbb{R}^n$:*

(i) *There exists convex sets $\tilde{\Omega}_k$, $k\in I$, where I is some countable index set, such that $\Omega=\bigcup_{k\in I}\tilde{\Omega}_k$.*

(ii) *Each point* $x \in \Omega$ *is in at most* N_1 *sets* $\tilde{\Omega}_k$.
(iii) *There exist* $t > 0, 0 < c_1 < c_2 < \infty$, *such that each* $\tilde{\Omega}_k$ *contains an Euclidean ball radius* $\geq c_1 t$ *and is contained in an Euclidean ball of radius* $\leq c_2 t$.

Then, for $f \in L_p(\Omega), 0 < p < \infty$

$$\sum_{k \in I} \omega_r\left(f, \tilde{\Omega}_k\right)_p^p \leq C(n, r, p, N_1, c_1, c_2)\, \omega_r(f, t)_{L_p(\Omega)}^p, \tag{4.8}$$

and for $p = \infty$

$$\sup_{k \in I} \omega_r\left(f, \tilde{\Omega}_k\right)_\infty \leq C(n, r, p, c_2)\, \omega_r(f, t)_{L_\infty(\Omega)}. \tag{4.9}$$

Theorem 4.6. *Let* Θ *be a cover of ellipsoids in* $\mathbb{R}^n$ *that are equivalent to Euclidean balls with fixed parameters. Then,* $\omega_{\Theta,r}\left(\cdot, 2^{-mn}\right)_p \sim \omega_r\left(\cdot, 2^{-m}\right)_p$ *where* $\omega_r(\cdot,\cdot)_p$ *is the classic isotropic modulus of smoothness defined in (3.12).*

Proof. From the properties of discrete covers, there exists $J(\Theta, r) > 0$ such that for each $m \in \mathbb{Z}$ and every $x \in \mathbb{R}^n$ there exists an ellipsoid $\theta \in \Theta_{mn-J}$ such that $\left\{y \in \mathbb{R}^n : |x - y| < r2^{-m}\right\} \subset \theta$. For each $\theta \in \Theta_{mn-J}$, denote by $X(\theta)$ the set of points $x \in \theta$, for which $\left\{y \in \mathbb{R}^n : |x - y| < r2^{-m}\right\} \subset \theta$. Since $\mathbb{R}^n = \bigcup_{\theta \in \Theta_{mn-J}} X(\theta)$ and each set $X(\theta)$ intersects with at most N_1 neighboring sets we get for $0 < p < \infty$,

$$\begin{aligned}
\omega_r\left(f, 2^{-m}, \mathbb{R}^n\right)_p^p &= \sup_{|h| \leq 2^{-m}} \int_{\mathbb{R}^n} \left|\Delta_h^r\left(f, x, \mathrm{R}^n\right)\right|^p dx \\
&\leq c \sup_{|h| \leq 2^{-m}} \sum_{\theta \in \Theta_{mn-J}} \int_{X(\theta)} \left|\Delta_h^r\left(f, x, \mathbb{R}^n\right)\right|^p dx \\
&\leq c \sum_{\theta \in \Theta_{mn-J}} \sup_{|h| \leq 2^{-m}} \int_\theta \left|\Delta_h^r(f, x, \theta)\right|^p dx \\
&= c \sum_{\theta \in \Theta_{mn-J}} \omega_r(f, \theta)_p^p \\
&= c\omega_{\Theta,r}\left(f, 2^{-(mn-J)}\right)_p^p \\
&\leq c\omega_{\Theta,r}\left(f, 2^{-mn}\right)_p^p,
\end{aligned}$$

where we applied (4.2) to obtain the last inequality. The case $p = \infty$ is similar and easier. Since we assume the ellipsoids are equivalent to Euclidean balls, they satisfy the conditions of Proposition 4.5, from which the inverse inequality is immediate. □

Next, we show a relationship between the anisotropic smoothness and isotropic smoothness under the following condition

Definition 4.7. We say that a cover Θ (satisfying the conditions of Definition 2.1 or Definition 2.2) is *quasi zero-uniform* if all the ellipsoids of the zero-level Θ_0 are equivalent in shape, i.e. for any $\theta, \theta' \in \Theta_0, c_1 \leq \left\|M_\theta^{-1} M_{\theta'}\right\| \leq c_2$, with the constants independent of the ellipsoids.

Observe that a quasi zero-uniform cover can still be highly anisotropic, since the shape of the ellipsoids can change as their level is finer. Similarly to Definition 2.6, we denote for $\mu := (\mu_0, \mu_1)$,

$$\mu(m) := \begin{cases} \mu_0, & m \le 0, \\ \mu_1, & m > 0. \end{cases} \qquad \tilde{\mu}(m) := \begin{cases} \mu_1, & m \le 0, \\ \mu_0, & m > 0. \end{cases} \tag{4.10}$$

Theorem 4.8. *Let Θ be a quasi zero-uniform cover and $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$. Then for $\mu = (a_6, a_4)$ and any $m \in \mathbb{Z}$,*

$$c\omega_r\left(f, 2^{-m\mu(m)}\right)_p \le \omega_{\Theta,r}\left(f, 2^{-m}\right)_p \le c\omega_r\left(f, 2^{-m\tilde{\mu}(m)}\right)_p, \tag{4.11}$$

where the parameters $0 < a_6 \le a_4 < \infty$, are from (2.2).

Obviously, (4.11) agrees with Theorem 4.6, whenever the ellipsoids of the cover are equivalent to Euclidean balls, since in such a case, one may choose $a_4 = a_6 = 1/n$.

For the proof of Theorem 4.8, it is convenient to use the machinery of K-functionals. Let $W_p^r(\Omega)$, $1 \le p \le \infty$, $r \in \mathrm{N}$, denote the *Sobolev spaces* [1], namely, the spaces of functions $g \in L_p(\Omega)$ which have all their distributional derivatives of order up to r in $L_p(\Omega)$. The semi-norm of $W_p^r(\Omega)$ is given by $|g|_{W_p^r(\Omega)} := \sum_{|\alpha|=r} \|D^\alpha g\|_{L_p(\Omega)} < \infty$. The *K-functional* of order r of $f \in L_p(\Omega)$, $1 \le p \le \infty$, is defined by

$$K_r(f,t)_{L_p(\Omega)} := K\left(f, t, L_p(\Omega), W_p^r(\Omega)\right) := \inf_{g \in W_p^r(\Omega)} \|f-g\|_{L_p(\Omega)} + t|g|_{W_p^r(\Omega)}.$$

The following equivalence for $\Omega = \mathbb{R}^n$, $1 \le p \le \infty$, is classic (e.g. Section 6 in [2], Section 5 in [9])

$$\omega_r(f,t)_{L_p(\mathbb{R}^n)} \sim K_r\left(f, t^r\right)_{L_p(\mathbb{R}^n)}, \quad \forall t > 0. \tag{4.12}$$

Also, in the special case $g \in W_p^r(\Omega)$, we have

$$\omega_r(g,t)_{L_p(\Omega)} \le ct^r |g|_{W_p^r(\Omega)}. \tag{4.13}$$

In the anisotropic case, we have the following variant

Lemma 4.9. *Let Θ be a cover, $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$ and $m \in \mathbb{Z}$. Then,*

(i) *For any $g \in W_p^r(\mathbb{R}^n)$*

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \le c \begin{cases} \|f-g\|_{L_p(\mathbb{R}^n)} + \left(\sum_{\theta \in \Theta_m} \operatorname{diam}(\theta)^{rp} |g|_{W_p^r(\theta)}^p\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{\theta \in \Theta_m} \left\{\|f-g\|_{L_\infty(\theta)} + \operatorname{diam}(\theta)^r |g|_{W_\infty^r(\theta)}\right\}, & p = \infty. \end{cases} \tag{4.14}$$

(ii) *For $T_m f := T_{m,p} f$, defined in (3.15),*

$$\omega_{\Theta,r}\left(f,2^{-m}\right)_p \ge c \begin{cases} \|f-T_m f\|_{L_p(\mathbb{R}^n)} + \left(\sum_{\theta\in\Theta_m} \text{width}(\theta)^{rp} |T_m f|^p_{W_p^r(\theta)}\right)^{1/p}, & 1\le p<\infty, \\ \sup_{\theta\in\Theta_m}\left\{\|f-T_m f\|_{L_\infty(\theta)} + \text{width}(\theta)^r |T_m f|_{W_\infty^r(\theta)}\right\}, & p=\infty. \end{cases} \tag{4.15}$$

Proof. To prove (4.14), for any $g\in W_p^r(\mathbb{R}^n)$, $1\le p<\infty$, we simply apply (4.13) on each ellipsoid $\theta\in\Theta_m$

$$\begin{aligned}\omega_{\Theta,r}\left(f,2^{-m}\right)_p^p &= \sum_{\theta\in\Theta_m}\omega_r(f,\theta)_p^p \\ &\le c\sum_{\theta\in\Theta_m}\left(\omega_r(f-g,\theta)_p^p+\omega_r(g,\theta)_p^p\right) \\ &\le c\sum_{\theta\in\Theta_m}\left(\|f-g\|^p_{L_p(\theta)}+\text{diam}(\theta)^{rp}|g|^p_{W_p^r(\theta)}\right) \\ &\le c\|f-g\|^p_{L_p(\mathbb{R}^n)}+\sum_{\theta\in\Theta_m}\text{diam}(\theta)^{rp}|g|^p_{W_p^r(\theta)}.\end{aligned}$$

We now prove (4.15) for $1\le p<\infty$. By (4.5), it is sufficient to prove that

$$\sum_{\theta\in\Theta_m}\text{width}(\theta)^{rp}|T_m f|^p_{W_p^r(\theta)} \le c\omega_{\Theta,r}\left(f,2^{-m}\right)_p^p. \tag{4.16}$$

Let $x\in\theta\in\Theta_m$. By the partition of unity of $\left\{\tilde{\phi}_{\theta'}\right\}$ (see (3.5)) we have

$$\begin{aligned}T_m f(x) &= \sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset} T_{\theta',p}(f)(x)\,\tilde{\phi}_{\theta'}(x) \\ &= T_{\theta,p}(f)(x) + \sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset}\left(T_{\theta',p}(f)(x)-T_{\theta,p}(f)(x)\right)\tilde{\phi}_{\theta'}(x).\end{aligned}$$

By property (e) of Definition 2.2, for any $\theta'\in\Theta_m$, such that $\theta\cap\theta'\neq\emptyset$, the intersection is ‘substantial’ and therefore, there exists a constant $\gamma(\Theta)\ge 1$, such that $|\theta|\le\gamma\left|\theta\cap\theta'\right|$. Thus, using (3.1), for any polynomial $P\in\Pi_{r-1}$,

$$\|P\|_{L_p(\theta)}\le c\|P\|_{L_p(\theta\cap\theta')}. \tag{4.17}$$

Condition (2.2) ensures that for any $\theta,\theta'\in\Theta_m$, $\theta\cap\theta'\neq\emptyset$, we have width $(\theta)\sim$ width $\left(\theta'\right)$. For any $\alpha\in\mathbb{Z}_+^n$, $|\alpha|=r$, we apply $\partial^\alpha T_{\theta,p}(f)=0$, the chain-rule, the polynomial norm estimate (4.17), the Markov inequality (3.3), and then (3.14) to obtain

$$\begin{aligned}\left\|\partial^\alpha T_m f\right\|^p_{L_p(\theta)} &\le c\sum_{\beta_1+\beta_2=\alpha,|\beta_2|\le r-1}\ \sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset}\left\|\partial^{\beta_1}\tilde{\phi}_{\theta'}\partial^{\beta_2}\left(T_{\theta',p}(f)-T_{\theta,p}(f)\right)\right\|^p_{L_p(\theta)} \\ &\le c\sum_{\beta_1+\beta_2=\alpha,|\beta_2|\le r-1}\ \sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset}\text{width}\left(\theta'\right)^{-|\beta_1|p} \\ &\quad\times\left\|\partial^{\beta_2}\left(T_{\theta',p}(f)-T_{\theta,p}(f)\right)\right\|^p_{L_p(\theta\cap\theta')}\end{aligned}$$

$$\leq c \sum_{\theta' \neq \theta, \theta' \cap \theta \neq \emptyset} \text{width}\left(\theta'\right)^{-rp} \left\| T_{\theta',p}(f) - T_{\theta,p}(f) \right\|_{L_p(\theta \cap \theta')}^p$$
$$\leq c \sum_{\theta' \cap \theta \neq \emptyset} \text{width}\left(\theta'\right)^{-rp} \left\| f - T_{\theta',p}(f) \right\|_{L_p(\theta')}^p$$
$$\leq c \sum_{\theta' \cap \theta \neq \emptyset} \text{width}\left(\theta'\right)^{-rp} \omega_r\left(f, \theta'\right)_p^p .$$

This implies

$$\text{width}(\theta)^{rp} \left| T_m f \right|_{W_p^r(\theta)}^p \leq c \sum_{\theta' \cap \theta \neq \emptyset} \omega_r\left(f, \theta'\right)_p^p ,$$

and as the number of ellipsoids intersecting θ is bounded by N_1, this yields (4.16) and concludes the proof of the lemma for $1 \leq p < \infty$. The proof for $p = \infty$, is similar and easier. □

Proof of Theorem 4.8. We prove the case $m > 0$. The case $m \leq 0$ is similar. The proof relies on the fact that in the quasi zero-uniform case, there exist constants, $0 < c_1 \leq c_2 < \infty$, such that for any ellipsoid $\theta' \in \Theta_m$, $m \geq 0$, we have

$$c_1 2^{-m a_4} \leq \text{width}\left(\theta'\right) \leq \text{diam}\left(\theta'\right) \leq c_2 2^{-m a_6}. \tag{4.18}$$

Indeed, for any $\theta' \in \Theta_m$, let $\theta \in \Theta_0$ such that $\theta \cap \theta' \neq \emptyset$. By the quasi zero-uniform assumption, $\|M_\theta\|, \left\|M_\theta^{-1}\right\| \leq c'$. Therefore, (2.2) yields

$$\text{diam}\left(\theta'\right) \leq 2 \left\| M_{\theta'} \right\| = 2 \left\| M_\theta M_\theta^{-1} M_{\theta'} \right\| \leq c \left\| M_\theta^{-1} M_{\theta'} \right\| \leq c a_5 2^{-m a_6}.$$

Similarly

$$\text{width}\left(\theta'\right) \geq \frac{1}{2} \left\| M_{\theta'}^{-1} \right\|^{-1} = \frac{1}{2} \left\| M_{\theta'}^{-1} M_\theta M_\theta^{-1} \right\|^{-1} \geq c \left\| M_{\theta'}^{-1} M_\theta \right\|^{-1} \geq c a_3 2^{-m a_4}.$$

To prove the right hand side of (4.11), for any $g \in W_p^r(\mathbb{R}^n)$, $1 \leq p < \infty$, we apply (4.14), (4.18) and then property (c) of Definition 2.2, to obtain

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \leq c \left(\|f - g\|_{L_p(\mathbb{R}^n)} + \left(\sum_{\theta \in \Theta_m} \text{diam}(\theta)^{rp} |g|_{W_p^r(\theta)}^p \right)^{1/p} \right)$$
$$\leq c \left(\|f - g\|_{L_p(\mathbb{R}^n)} + 2^{-m a_6 r} \left(\sum_{\theta \in \Theta_m} |g|_{W_p^r(\theta)}^p \right)^{1/p} \right)$$
$$\leq c \left(\|f - g\|_{L_p(\mathbb{R}^n)} + 2^{-m a_6 r} |g|_{W_p^r(\mathbb{R}^n)} \right).$$

Therefore, $\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \leq c K_r\left(f, 2^{-m a_6 r}\right)_p$ and by (4.12), $\omega_{\Theta,r}\left(f, 2^{-m}\right)_p \leq c \omega_r \left(f, 2^{-m a_6}\right)_p$.

The proof of the left hand side of (4.11) for $1 \leq p < \infty$, is by application of (4.12), then (4.18) and finally (4.15)

$$\omega_r\left(f, 2^{-m a_4}\right)_p \leq c K_r\left(f, 2^{-m a_4 r}\right)_p$$
$$\leq c \left(\|f - T_m f\|_{L_p(\mathbb{R}^n)} + 2^{-m a_4 r} |T_m f|_{W_p^r(\mathbb{R}^n)} \right)$$

$$\leq c\left(\|f-T_m f\|_{L_p(\mathbb{R}^n)}+\left(\sum_{\theta\in\Theta_m}\text{width}\,(\theta)^{rp}\,|T_m f|^p_{W^r_p(\theta)}\right)^{1/p}\right)$$
$$\leq c\omega_{\Theta,r}\left(f,2^{-m}\right)_p .$$

The proof for the case $p=\infty$ is similar. □

5. Ul'yanov-type inequality

The classic Ul'yanov inequality [13] for periodic functions $f \in L_p(\mathbb{T})$, $1 \leq p \leq q < \infty$, is

$$\omega(f,t)_q \leq c\left(\int_0^t \left(u^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\omega(f,u)_p\right)^q \frac{du}{u}\right)^{1/q}.$$

An higher order (but slightly weaker) version for $f \in L_p(\mathbb{R})$, $1 \leq p \leq q < \infty$, is (see Theorem 3.4, Chapter 6 in [9])

$$\omega_r(f,t)_q \leq c\int_0^t u^{-\left(\frac{1}{p}-\frac{1}{q}\right)}\omega_r(f,u)_p \frac{du}{u}.$$

Following the methods of [10], who proved a sharp result for the 'full range' of indices, we show a discrete, anisotropic analogue

Theorem 5.1. *For a cover Θ, $f \in L_p(\mathbb{R}^n)$, $0 < p \leq q \leq \infty$, and any $m \in \mathbb{Z}$,*

$$\omega_{\Theta,r}\left(f,2^{-m}\right)_q \leq c\left(\sum_{j=m}^{\infty} 2^{j\left(\frac{1}{p}-\frac{1}{q}\right)\gamma}\omega_{\Theta,r}\left(f,2^{-j}\right)_p^{\gamma}\right)^{1/\gamma}, \tag{5.1}$$

where

$$\gamma := \begin{cases} q, & 0<q<\infty,\\ 1, & q=\infty.\end{cases}$$

To prove Theorem 5.1, we need a few results, the first of which is the following Nikol'skii-type estimate

Theorem 5.2. *For $f \in L_p^{\text{loc}}(\mathbb{R}^n)$, $0 < p \leq q \leq \infty$, and $m \in \mathbb{Z}$,*

$$\|T_{m+1}f - T_m f\|_q \leq c2^{m\left(\frac{1}{p}-\frac{1}{q}\right)}\omega_{\Theta,r}\left(f,2^{-m}\right)_p . \tag{5.2}$$

Remark. Here, and throughout this section, $T_m f := T_{m,p} f$, i.e. the approximation associated with the p-norm.

Proof. Since $T_{m+1}f - T_m f \in W_m$ (see Section 3.4), there exists a representation $T_{m+1}f - T_m f = \sum_{\nu\in\mathcal{M}_m} a_\nu F_\nu$. Applying (3.17) for the q-norm, $q < \infty$, the assumption $p \leq q$, then (3.17) again for the p-norm and finally (4.5) give

$$\|T_{m+1}f - T_m f\|_q \leq c2^{m\left(\frac{1}{2}-\frac{1}{q}\right)}\left(\sum_{\nu\in\mathcal{M}_m}|a_\nu|^q\right)^{1/q}$$

$$\le c 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)} 2^{m\left(\frac{1}{2}-\frac{1}{p}\right)} \left(\sum_{\nu\in\mathcal{M}_m} |a_\nu|^p\right)^{1/p}$$
$$\le c 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)} \|T_{m+1}f - T_m f\|_p$$
$$\le c 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_{\Theta,r}\left(f, 2^{-m}\right)_p .$$

The proof for the case $q = \infty$ is similar. □

Lemma 5.3. *For $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and $0 < p \le q \le \infty$,*

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_q \le c\left(\|f - T_m f\|_q + 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_{\Theta,r}\left(f, 2^{-m}\right)_p\right). \tag{5.3}$$

Proof. First observe that

$$\omega_{\Theta,r}\left(f, 2^{-m}\right)_q \le c\left(\omega_{\Theta,r}\left(f - T_m f, 2^{-m}\right)_q + \omega_{\Theta,r}\left(T_m f, 2^{-m}\right)_q\right).$$

Since by Theorem 4.2

$$\omega_{\Theta,r}\left(f - T_m f, 2^{-m}\right)_q \le c\,\|f - T_m f\|_q ,$$

it suffices to show that

$$\omega_{\Theta,r}\left(T_m f, 2^{-m}\right)_q \le c 2^{m\left(\frac{1}{p}-\frac{1}{q}\right)} \omega_{\Theta,r}\left(f, 2^{-m}\right)_p .$$

By definition, for $0 < q < \infty$,

$$\omega_{\Theta,r}\left(T_m f, 2^{-m}\right)_q^q = \sum_{\theta\in\Theta_m} \omega_r\left(T_m f, \theta\right)_q^q .$$

By the partition of unity of $\left\{\tilde{\phi}_{\theta'}\right\}$

$$\omega_r\left(T_m f, \theta\right)_q^q = \sup_{h\in\mathbb{R}^n} \left\| \Delta_h^r \left(\sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset} T_{\theta',p}(f)\,\tilde{\phi}_{\theta'}\right)\right\|_{L_q(\theta)}^q$$
$$= \sup_{h\in\mathbb{R}^n} \left\| \Delta_h^r \left(T_{\theta,p}(f) + \sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset} \left(T_{\theta',p}(f) - T_{\theta,p}(f)\right)\tilde{\phi}_{\theta'}\right)\right\|_{L_q(\theta)}^q$$
$$\le c \sum_{\theta'\neq\theta,\theta'\cap\theta\neq\emptyset} \left\|T_{\theta',p}(f) - T_{\theta,p}(f)\right\|_{L_q(\theta)}^q .$$

Using (4.17), then (3.2) and then (3.14) yields for each $\theta' \in \Theta_m, \theta' \neq \theta, \theta' \cap \theta \neq \emptyset$,

$$\left\|T_{\theta',p}(f) - T_{\theta,p}(f)\right\|_{L_q(\theta)}^q \le c\left\|T_{\theta',p}(f) - T_{\theta,p}(f)\right\|_{L_q(\theta\cap\theta')}^q$$
$$\le c 2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|T_{\theta',p}(f) - T_{\theta,p}(f)\right\|_{L_p(\theta\cap\theta')}^q$$
$$\le c 2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\left\|f - T_{\theta,p}(f)\right\|_{L_p(\theta)}^q\right.$$

$$+\left\|f-T_{\theta',p}(f)\right\|_{L_p(\theta')}^q\Big)$$
$$\leq c2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\omega_r(f,\theta)_p^q+\omega_r\left(f,\theta'\right)_p^q\right).$$

We apply this and $q \geq p$ to obtain

$$\begin{aligned}\omega_{\Theta,r}\left(T_m f,2^{-m}\right)_q^q &\leq c2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)}\sum_{\theta\in\Theta_m}\omega_r(f,\theta)_p^q\\ &\leq c2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\sum_{\theta\in\Theta_m}\omega_r(f,\theta)_p^p\right)^{q/p}\\ &= c2^{mq\left(\frac{1}{p}-\frac{1}{q}\right)}\omega_{\Theta,r}\left(f,2^{-m}\right)_p^q.\end{aligned}$$

This concludes the proof of the lemma for $0 < q < \infty$. The proof for $q = \infty$ is similar. □

Proof of Theorem 5.1. By (5.3)

$$\omega_{\Theta,r}\left(f,2^{-m}\right)_q \leq c\left(\|f-T_m f\|_q+2^{m\left(\frac{1}{p}-\frac{1}{q}\right)}\omega_{\Theta,r}\left(f,2^{-m}\right)_p\right).$$

Let us replace for a moment the first right hand side term $\|f-T_m f\|_q$ by $\|T_M f-T_m f\|_q$ for a 'large' $M > m$. Then, for $0 < q \leq 1$ we have by (5.2)

$$\|T_M f-T_m f\|_q^q \leq \sum_{j=m}^{M-1}\left\|T_{j+1}f-T_j f\right\|_q^q \leq c\sum_{j=m}^{M-1}2^{j\left(\frac{1}{p}-\frac{1}{q}\right)q}\omega_{\Theta,r}\left(f,2^{-j}\right)_p^q.$$

For $1 \leq q \leq \infty$, we similarly get

$$\|T_M f-T_m f\|_q \leq \sum_{j=m}^{M-1}\left\|T_{j+1}f-T_j f\right\|_q \leq c\sum_{j=m}^{M-1}2^{j\left(\frac{1}{p}-\frac{1}{q}\right)}\omega_{\Theta,r}\left(f,2^{-j}\right)_q.$$

However, note that for $1 < q < \infty$, we claim a sharper estimate in (5.1), using the l_q-norm of $\left\{2^{j(1/p-1/q)}\omega_{\Theta,r}\left(f,2^{-j}\right)_q\right\}$ instead of the l_1-norm. Indeed, this is achieved using exactly the proof of Lemma 3.1 in [10], which requires the Nikol'skii-type estimate (5.2) and gives, for $1 < q < \infty$,

$$\|T_M f-T_m f\|_q \leq c\left(\sum_{j=m}^{M-1}2^{j\left(\frac{1}{p}-\frac{1}{q}\right)q}\omega_{\Theta,r}\left(f,2^{-j}\right)_q^q\right)^{1/q}.$$

Therefore, to prove (5.1) it remains to show that if the right hand side of (5.1) is finite, then

$$\|T_M f-T_m f\|_q \underset{M\to\infty}{\to} \|f-T_m f\|_q. \tag{5.4}$$

Observe that for any $j \in \mathbb{Z}$, $T_j f = T_{j,p} f \in L_q(\mathbb{R}^n)$, since using (3.11) with $q \geq p$,

$$\left\|T_j f\right\|_q \leq c_1 2^{j\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|T_j f\right\|_p \leq c_2 2^{j\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p.$$

Therefore, if the right hand side of (5.1) is finite, then $\{T_M f\}$ is a Cauchy sequence in $L_q(\mathbb{R}^n)$ and thus converges in the measure to a function F. But since $\{T_M f\}$ converge in $L_p(\mathbb{R}^n)$ to f and thus also in the measure to f, we have that $F = f$ a.e. and therefore (5.4) is proved. □

6. Embeddings of anisotropic Besov spaces

We briefly recall from [4] the homogeneous B-spaces $\dot{B}^{\alpha}_{pq}(\Theta)$ induced by an arbitrary discrete ellipsoid cover Θ, with $0 < p, q \le \infty$ and smoothness index $\alpha > 0$. For $f \in L^{\mathrm{loc}}_{p}(\mathbb{R}^n)$ we define

$$\|f\|_{\dot{B}^{\alpha}_{pq}(\Theta)} := \begin{cases} \left(\sum_{m\in\mathbb{Z}} \left(2^{\alpha m/n} \omega_{\Theta,r}\left(f, 2^{-m}\right)_p \right)^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} 2^{\alpha m/n} \omega_{\Theta,r}\left(f, 2^{-m}\right)_p, & q = \infty, \end{cases} \tag{6.5}$$

where $r \ge 1$ satisfies

$$r > \frac{1}{a_6} \cdot \frac{\alpha}{n}, \tag{6.6}$$

the $\omega_{\Theta,r}(\cdot,\cdot)_p$ is the anisotropic modulus of smoothness (4.1) and a_6 is defined in (2.2). It is easy to see that definition (6.5) is equivalent to the one given in [4]

$$\|f\|_{\dot{B}^{\alpha}_{pq}(\Theta)} \sim \left(\sum_{m\in\mathbb{Z}} \left(\sum_{\theta\in\Theta_m} |\theta|^{-\alpha p/n} \omega_r(f,\theta)_p^p \right)^{q/p} \right)^{1/q},$$

for $0 < p, q < \infty$, with the obvious modifications for p or $q = \infty$.

Observe that $\dot{B}^{\alpha}_{pq}(\Theta)$ is actually a quotient space modulo Π_{r-1} (see the detailed discussion in Section 5 of [4]). By applying the anisotropic Marchaud inequality (4.6) and the discrete Hardy inequality, it can be shown, using the same method of proof as in the isotropic case (see Lemma 3.4, Section 2 in [9]), that the norms (6.5) are equivalent for different values of r satisfying (6.6). Also, by Theorem 4.6, we have that in the special case where all the ellipsoids of the cover are equivalent to Euclidean balls, $\dot{B}^{\alpha}_{pq}(\Theta)$ is equivalent to $\dot{B}^{\alpha}_{pq}(\mathbb{R}^n)$, the classic (isotropic) Besov space.

Using the operators $\{T_m\}$, $m \in \mathbb{Z}$, from Section 3.3 and the 'two-level splits' from Section 3.4, we define

$$\|f\|^{T}_{\dot{B}^{\alpha}_{pq}(\Theta)} := \begin{cases} \left(\sum_{m\in\mathbb{Z}} \left(2^{\alpha m/n} \|(T_{m+1} - T_m) f\|_p \right)^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{m\in\mathbb{Z}} 2^{\alpha m/n} \|(T_{m+1} - T_m) f\|_p, & q = \infty. \end{cases}$$

$$\|f\|^{A}_{\dot{B}^{\alpha}_{pq}(\Theta)} := \inf_{f = \sum_{\nu\in\mathcal{M}} a_\nu F_\nu} \left(\sum_{m\in\mathbb{Z}} \left(\sum_{\nu\in\mathcal{M}_m} \left(|\eta_\nu|^{-\alpha/n} \|a_\nu F_\nu\|_p \right)^p \right)^{q/p} \right)^{1/q}.$$

Theorem 6.1 (*Theorem 5.8 in [4]*)**.** *For a discrete cover Θ, $0 < p, q \le \infty$ and $\alpha > 0$, if (6.6) is obeyed, then the norms $\|\cdot\|_{\dot{B}^{\alpha}_{pq}(\Theta)}$, $\|\cdot\|^{T}_{\dot{B}^{\alpha}_{pq}(\Theta)}$ and $\|\cdot\|^{A}_{\dot{B}^{\alpha}_{pq}(\Theta)}$ are equivalent.*

As in the isotropic case, the Ul'yanov inequality can be applied to obtain embedding results for the anisotropic Besov spaces.

Theorem 6.2. *Let Θ be a cover of $\mathbb{R}^n$, $0 < p < q \le \infty$, and denote $\gamma := 1/p - 1/q$. Then, for $\alpha > 0$, the following (continuous) embedding hold*

(i) $\dot{B}^{\alpha+\gamma n}_{p,\infty}(\Theta) \subset \dot{B}^{\alpha}_{q,\infty}(\Theta)$.

(ii) $\dot{B}^{\alpha+\gamma n}_{p,q}(\Theta) \subset \dot{B}^{\alpha}_{q,q}(\Theta)$.

Proof. (i) Let $f \in \dot{B}^{\alpha+\gamma n}_{p,\infty}(\Theta)$. For $q < \infty$, and any $m \in \mathbb{Z}$, we have by (5.1)

$$\begin{aligned}\omega_{\Theta,r}\left(f, 2^{-m}\right)_q^q &\le c \sum_{j=m}^{\infty} 2^{j\gamma q} \omega_{\Theta,r}\left(f, 2^{-j}\right)_p^q \\ &\le c\, |f|^q_{\dot{B}^{\alpha+\gamma n}_{p,\infty}(\Theta)} \sum_{j=m}^{\infty} 2^{j\gamma q} 2^{-j(\alpha/n+\gamma)q} \\ &\le c\, |f|^q_{\dot{B}^{\alpha+\gamma n}_{p,\infty}(\Theta)} 2^{-m\alpha q/n}.\end{aligned}$$

The proof for $q = \infty$ is similar.

(ii) For $0 < q < \infty$, the Ul'yanov inequality (5.1) gives

$$\begin{aligned}|f|^q_{\dot{B}^{\alpha}_{q,q}(\Theta)} &= \sum_m \left(2^{m\alpha/n} \omega_{\Theta,r}\left(f, 2^{-m}\right)_q\right)^q \\ &\le c \sum_m \sum_{j=m}^{\infty} 2^{m\alpha q/n} 2^{j\gamma q} \omega_{\Theta,r}\left(f, 2^{-j}\right)_p^q \\ &= c \sum_j 2^{j\gamma q} \omega_{\Theta,r}\left(f, 2^{-j}\right)_p^q \sum_{m=-\infty}^{j} 2^{m\alpha q/n} \\ &= c \sum_j 2^{jq(\alpha+\gamma n)/n} \omega_{\Theta,r}\left(f, 2^{-j}\right)_p^q \sum_{m=-\infty}^{j} 2^{(m-j)\alpha q/n} \\ &\le c \sum_j 2^{jq(\alpha+\gamma n)/n} \omega_{\Theta,r}\left(f, 2^{-j}\right)_p^q \\ &\le c\, |f|^q_{\dot{B}^{\alpha+\gamma n}_{p,q}(\Theta)}. \quad \square\end{aligned}$$

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