Mathematical Foundations of Machine Learning, Spring 2024: Assignment I

1. [30%] Let $f(x) := \sum_{m=1}^{M} c_m \mathbf{1}_{[2m,2m+1]}(x)$. Compute the modulus $\omega_1(f,t)_p$, for all $0 < t < 1/2$, and $0 < p \le \infty$.

Remark: There are two cases: $0 < p < \infty$, $p = \infty$.

2. [30%] Prove the following equality for any $N \ge 1$, $x, h \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta_{Nh}^{r}(f,x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta_h^r(f, x + k_1 h + \ldots + k_r h).$$

Hint: recall we proved in class for $r = 1$. Now apply induction on r. <u>Make sure the notations are correct</u>.

3. [30%] Let $f : [0,1]^n \to \mathbb{R}^L$ and let $\Omega \subseteq [0,1]^n$. Prove that minimizing the variance over partitions $\Omega' \cup \Omega'' = \Omega$,

$$V_\Omega := \int_{\Omega'} \left| \vec{f}(x) - \vec{E}_{\Omega'} \right|_{l_2(\mathbb{R}^L)}^2 dx + \int_{\Omega''} \left| \vec{f}(x) - \vec{E}_{\Omega''} \right|_{l_2(\mathbb{R}^L)}^2 dx,$$

is equivalent to maximizing the wavelet norms

$$\|\psi_{\Omega'}\|^2 + \|\psi_{\Omega''}\|^2,$$

where $\vec{E}_{\Omega'} = \frac{1}{|\Omega'|} \int_{\Omega'} \vec{f}(x) dx$, $\quad \|\psi_{\Omega'}\|_2 = |\Omega'|^{1/2} \left| \vec{E}_{\Omega'} - \vec{E}_{\Omega} \right|_{l_2(\mathbb{R}^L)}$.

4. [10%] How would you speed up the training of a Random Forest composed of 5 trees using 20 parallel processors? Try to describe an optimal scenario where the 20 processors are fully utilized all the time.