Foundations of approximation theory: Assignment I

1. [Minkowski integral inequality] Prove that for $1 \le p < \infty$ and a measurable function $F(x,t): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$\left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |F(x,t)| dx \right)^p dt \right)^{1/p} \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |F(x,t)|^p dt \right)^{1/p} dx .$$

Hints: for $1 < p < \infty$ $\left(\int_{\mathbb{R}^n} |F(x,t)| dx \right)^p = \left(\int_{\mathbb{R}^n} |F(x,t)| dx \right) \left(\int_{\mathbb{R}^n} |F(y,t)| dy \right)^{p-1}$, change order of integration of t and x, use Hölder.

2. Recall that a function $g \in L_1(\mathbb{R}^n)$ is the **distributional derivative** of $f \in L_1(\mathbb{R}^n)$, $g := \partial^\alpha f$, $\alpha \in \mathbb{Z}_+^n$, if

$$\int_{\mathbb{R}^n} g\phi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f \partial^\alpha \phi \ , \qquad \forall \phi \in C_0^\infty(\mathbb{R}^n) \ .$$

Prove $H'(x) = \begin{cases} \overline{1, \quad -1 \le x < 0,} \\ -1, \quad 0 \le x \le 1, \\ 0, \quad \textit{else.} \end{cases}$ where $H(x) := \begin{cases} \overline{x+1, \quad -1 \le x < 0,} \\ 1-x, \quad 0 \le x \le 1, \\ 0, \quad \textit{else.} \end{cases}$

3. [Convergence of Fourier series] Compute the Fourier coefficients $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$ for:

   a. $f(x) = x$,

   b. $f(x) = x^2$.

   Recall that, by Parseval, the degree of approximation of the Fourier series is

$$E_N(f)_2 := \| f - S_N f \|_2 = \left(\sum_{|k|>N} \left| \hat{f}(k) \right|^2 \right)^{1/2} .$$

Estimate the error for the above two cases as a function of N. What is the reason for the qualitative difference in the rate of decay of the error (as $N \to \infty$) for these two examples?

4. [Partial Fourier series]

   a. Let $f \in L_2[-\pi,\pi]$ be 2π-periodic, with $\|f\|_2 \le 1$. Is it true that $\|S_N f\|_2 \le 1$, $N \ge 0$?

   b. Let $f \in L_\infty[-\pi,\pi]$ be 2π-periodic, with $\|f\|_\infty \le 1$. Is it true that $\|S_N f\|_\infty \le 1$, $N \ge 0$?

5. For $\phi \in L_2(\mathbb{R}^n)$, $\left[\hat{\phi},\hat{\phi}\right](w) := \sum_{k \in \mathbb{Z}^n} \left| \hat{\phi}(w + 2\pi k) \right|^2 \in L_2\left([-\pi,\pi]^n \right)$ is called the auto-correlation function.

   Prove that $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}^n}$ are an orthonormal system iff $\left[\hat{\phi},\hat{\phi}\right](w) = 1$, a. e.

Hint: $\langle \phi, \phi(\cdot + j) \rangle = (2\pi)^{-n} \left\langle \phi^\wedge, (\phi(\cdot + j))^\wedge \right\rangle = \ldots$ the Fourier coefficients of $\left[\hat{\phi},\hat{\phi}\right]$.

6. Let $f(x) := \sum_{m=1}^{M} c_m \mathbf{1}_{[2m,2m+1]}(x)$. Compute the modulus $\omega_1(f,t)_p$, for all $0 < t < 1/2$, and $0 < p \le \infty$.

7. Prove the following equality for any $N \ge 1$, $x,h \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta_{Nh}^r(f,x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta_h^r(f, x + k_1 h + \ldots + k_r h).$$

Hint: recall we proved in class for $r = 1$. Now apply induction on r. <u>Make sure the notations are correct</u>.

8. Recall that we proved in class for $g \in C^r(\mathbb{R}) \cap W_p^r(\mathbb{R})$, $1 \le p \le \infty$, that

$$\omega_r(g,t)_{L_p(\Omega)} \le C(r,n) t^r |g|_{W_p^r(\Omega)}, \qquad \forall t > 0.$$

Complete the proof for a general $g \in W_p^r(\mathbb{R})$, $1 \le p < \infty$ by using a ‘density’ argument, .i.e a sequence of functions

$$\{g_k\} \subset C^r(\mathbb{R}) \cap W_p^r(\mathbb{R}), \qquad \|g_k - g\|_{W_p^r(\mathbb{R})} \underset{k\to\infty}{\to} 0.$$