Analysis of spaces of homogenous type 2016:

Theorem list for the exam

General information

A. In cases where the setting is a general 'geometric' space X, we assume the space is a space of homogeneous type equipped with a quasi-distance ρ and a measure $\mu(E)=|E|$, that satisfy:

(i) $\exists c_1>1$ such that $\forall x,y\in M$ and $\delta>0$, $B(x,\delta)\cap B(y,\delta)\neq\emptyset \Rightarrow B(y,\delta)\subseteq B(x,c_1\delta)$,

(ii) $\exists c_2>1$ such that $|B(x,c_1 r)|\le c_2|B(x,r)|$, $\forall x\in X$, $r>0$.

B. In cases where the setting is a 'manifold' M, we assume that ρ is a distance and that the doubling condition gives $|B(x,\lambda r)|\le c_0\lambda^d|B(x,r)|$, $\lambda\ge 1$. Also, recall the notation

$$D_{\delta,\sigma}(x,y):=\left(|B(x,\delta)||B(y,\delta)|\right)^{-1/2}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}.$$

Theorems for 17 points

1. Recall that a summability kernel sequence $\{h_N\}_{N\in\mathbb{N}}$ over $[-\pi,\pi]$ satisfies the following conditions

   A. $\frac{1}{2\pi}\int_{-\pi}^{\pi}h_N(x)\,dx=1$, $\forall N\in\mathbb{N}$,

   B. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|h_N(x)|\,dx\le c$, $\forall N\in\mathbb{N}$,

   C. For any $0<\delta<\pi$, $\lim_{N\to\infty}\frac{1}{2\pi}\int_{|x|>\delta}|h_N(x)|\,dx=0$.

   Prove that for any $f\in C[-\pi,\pi]$

$$\|f-f*h_N\|_\infty \underset{N\to\infty}{\to} 0.$$

2. Prove the Vitali Covering Lemma. If $E\subset X$ is a finite union of balls, then there exists a pairwise disjoint subset $\tilde{B}_1,\dots,\tilde{B}_m$, such that

$$\sum_{k=1}^{m}|\tilde{B}_k|\ge c_2^{-1}|E|.$$

3. Let $\phi\in C(\mathbb{R})$ be even, $\operatorname{supp}(\hat{\phi})\subseteq[-A,A]$, for some $A>0$ and $\hat{\phi}\in C^2(\mathbb{R})$. Then for $\delta>0$

$$\phi(\delta\sqrt{L})(x,y)=0, \text{ if } \rho(x,y)>\tilde{c}\delta A, \quad \forall x,y\in M.$$

   Remarks:

   (i) Here, one may assume that ρ is a distance. i.e. $\rho(x,y)\le\rho(x,z)+\rho(z,y)$.

(ii) One may assume the Finite Propagation property for the operator $\cos\left(t\sqrt{L}\right)$, $t>0$.

4. Let $1\le p\le\infty$ and let $m\in\mathbb{N}$. Then, there exists $c_m>0$ such that for any $f\in\Sigma_\lambda^p(L,M)$, $\lambda\ge1$,

$$\left\|L^m f\right\|_p\le c_m\lambda^{2m}\|f\|_p.$$

Remark: Please quote precisely the theorems you are using for the proof.

5. The Sinc function is defined by $\phi(x):=\mathcal{F}^{-1}\left(\mathbf{1}_{[-\pi,\pi]^n}(\cdot)\right)(x)$. $S(\phi):=\overline{\operatorname{span}}\{\phi(\cdot-k)\;:k\in\mathbb{Z}^n\}$. Prove that for any $r\ge1$, $h>0$ and $g\in W_2^r(\mathbb{R}^n)$

$$E\left(g,S(\phi)^h\right)_2\le ch^r|g|_{r,2}.$$

Remark: Here $g\in W_2^r(\mathbb{R}^n)$ means that $\exists\partial^\alpha g$, $\forall\alpha\in\mathbb{Z}^n$, $|\alpha|\le r$ and $\partial^\alpha g\in C(\mathbb{R}^n)\cap L_2(\mathbb{R}^n)$.

6. Let $1\le p\le\infty$ and let $m\in\mathbb{N}$. Then, there exists $c_m>0$ such that for any $t\ge1$

$$E_t(f)_p:=\inf_{g\in\Sigma_t^p}\|f-g\|\le c_m t^{-2m}\left\|L^m f\right\|_p,\qquad f\in D\left(L^m\right)\cap L_p(M).$$

7. Prove that if $\phi\in\mathcal{S}(\mathbb{R})$ is even, then for any $k\ge1$ and $\delta>0$, the kernel of $\phi\left(\delta\sqrt{L}\right)$ satisfies

$$\left|\phi\left(\delta\sqrt{L}\right)(x,y)\right|\le c_k D_{\delta,k}(x,y).$$

Remark: You should use the estimate for univariate even $C^k(\mathbb{R})$ functions with support in $[-R,R]$ (Theorem 3.1 in [KP]).

Theorems for 34 Points

8. [Schwartz class]

A. Show that the following two definitions for the Schwartz class are equivalent:

(i) $\phi\in\mathcal{S}(\mathbb{R}^n)$ if $\phi\in C^\infty(\mathbb{R}^n)$ and for any multi-indices $\alpha,\beta\in\mathbb{Z}_+^n$,

$$\rho_{\alpha,\beta}(\phi):=\sup_{x\in\mathbb{R}^n}\left|x^\alpha\partial^\beta\phi(x)\right|=C_{\alpha,\beta}<\infty.$$

(ii) $\phi\in\mathcal{S}(\mathbb{R}^n)$ if $\phi\in C^\infty(\mathbb{R}^n)$ and for any multi-index $\alpha\in\mathbb{Z}_+^n$ and integer $N\ge1$

$$\sup_{x\in\mathbb{R}^n}\left|\partial^\alpha\phi(x)\right|\left(1+|x|\right)^N\le C_{\alpha,N}<\infty.$$

B. Prove that if $\phi \in \mathcal{S}(\mathbb{R})$ then $\hat{\phi} \in \mathcal{S}(\mathbb{R})$ by estimating the Schwartz norms

$\rho_{\alpha,\beta}(\hat{\phi}) := \sup_{w\in\mathbb{R}} \left| w^{\alpha} \hat{\phi}^{(\beta)}(w) \right|$.

9. Prove the following for $\sigma > d$:

A. For any $x \in M$ and $\delta > 0$

$$\int_M \frac{1}{\left(1+\delta^{-1}\rho(x,u)\right)^{\sigma}} d\mu(u) \le c\left|B(x,\delta)\right|.$$

B. For any $x, y \in M$ and $\delta > 0$

$$\int_M \frac{1}{\left(1+\delta^{-1}\rho(x,u)\right)^{\sigma}\left(1+\delta^{-1}\rho(y,u)\right)^{\sigma}} d\mu(u) \le c\frac{\left|B(x,\delta)\right| + \left|B(y,\delta)\right|}{\left(1+\delta^{-1}\rho(x,y)\right)^{\sigma}} \le \tilde{c}\frac{\left|B(x,\delta)\right|}{\left(1+\delta^{-1}\rho(x,y)\right)^{\sigma-d}}.$$

10. Let Θ be a discrete ellipsoid multi-level cover of $\mathbb{R}^n$. Define $\rho(x,y) := \inf_{\theta\in\Theta}\left\{|\theta| : x, y \in \theta\right\}$. Prove that $\exists \kappa \ge 1$, depending only on the parameters of the cover, such that for any $x, y, z \in \mathbb{R}^n$

$$\rho(x,y) \le \kappa\left(\rho(x,z) + \rho(z,y)\right).$$

Remark: One needs to **prove** (and then apply) the Lemma that states that there exists a constant $s^* \ge 1$, such that for any $\theta \in \Theta_m$, $\theta' \in \Theta_{m+v}$, $v \ge 0$, $\theta \cap \theta' \ne \emptyset$, there exists $\eta \in \Theta_{m-s^*}$, for which $\theta \cup \theta' \subset \eta$.

11. Let $0 < p \le \infty$ and let $g \in \Sigma_{\lambda}^{p}(M, L)$, $\lambda \ge 1$.

A. Prove that

$$\|g\|_{\infty} \le c\left\| \left|B(\cdot,\lambda^{-1})\right|^{-1/p} g(\cdot) \right\|_p.$$

B. Assuming the non-collapsing condition $\inf_{x\in M}\left|B(x,1)\right| \ge c$, prove that

$$\|g\|_{\infty} \le c\lambda^{d/p}\|g\|_p.$$

12. Recall the centered and un-centered Maximal functions. For any measurable function f

$$Mf(x) := \sup_{r>0}\frac{1}{\left|B(x,r)\right|}\int_{B(x,r)}\left|f(y)\right|d\mu(y), \qquad \tilde{M}f(x) := \sup_{x\in B}\frac{1}{|B|}\int_B\left|f(y)\right|d\mu(y).$$

A. Prove that if $f \in L_1$, then $\|Mf\|_{1,\infty} := \sup_{\alpha>0} \alpha \left|\{x \in M : |f(x)| > \alpha\}\right| \le c\|f\|_1$.

B. Prove that if $f \in L_p$, $1 < p \le \infty$, then $\|Mf\|_p \le c\|f\|_p$.

Remark: You may use the equality for $g \in L_p(M)$, $0 < p < \infty$

$$\|g\|_p^p = p\int_0^\infty \left|\{x : |g(x)| > \alpha\}\right| \alpha^{p-1} d\alpha .$$

13. Prove that for any $0 < p \le 1$, $H_a^p(\mathbb{R}^n) \subseteq H^p(\mathbb{R}^n)$, where $H_a^p(\mathbb{R}^n)$ is the 'Atomic' Hardy space.

Remarks:

A. You need to prove that for any (p,∞)-atom a, $\|a\|_{H^p(\mathbb{R}^n)} \le c$.

B. You may assume the quasi-triangle inequality, i.e. for any $\{f_k\}$, $f_k \in L_p(\mathbb{R}^n)$, $k \ge 1$,

$$\left\|\sum_j f_j\right\|_p^p \le \sum_j \|f_j\|_p^p .$$

14. Prove that for any real trigonometric polynomial P of degree N

$$P'(t)^2 + N^2 P(t)^2 \le N^2 \|P\|_\infty^2 \;,\; t \in [-\pi,\pi].$$

15. Prove the Davis-Gaffney estimate

$$\left|\langle P_t f_1, f_2\rangle\right| \le \exp\left\{-\frac{c^* r^2}{t}\right\} \|f_1\|_2 \|f_2\|_2 \;,\; t > 0,$$

where $\operatorname{supp}(f_1) \subseteq U_1$, $\operatorname{supp}(f_2) \subseteq U_2$ and $r := \rho(U_1,U_2) > 0$.

Remarks:

(i) You may assume that the function $F(z) := \langle P_z f_1, f_2\rangle$, is holomorphic in $\mathbb{C}_+$ and that $|F(z)| \le \|f_1\|_2 \|f_2\|_2$.

(ii) You may use the Phragmén-Lindoelöf theorem that states that for an holomorphic F in $\mathbb{C}_+$ with $|F(z)| \le B$, $z \in \mathbb{C}_+$ and $|F(t)| \le Ae^{-\gamma/t}$, $t > 0$, we have $|F(z)| \le Be^{-\operatorname{Re}(\gamma/z)}$.

(iii) You need to prove stronger ε-version of the Phragmén-Lindoelöf theorem .

16. Prove that for an even $f \in C^k(\mathbb{R})$, $k \ge d+1$, with $\operatorname{supp}(f) \subseteq [-R,R]$, for some $R \ge 1$

$$\left| f\left(\delta\sqrt{L}\right)(x,y)\right| \le c_k D_{\delta,k}(x,y),$$

where $c_k = c_k(f) = R^d\left[c_1(k)\|f\|_\infty + c_2(k)R^k\left\|f^{(k)}\right\|_\infty\right]$.

Remarks:

(i) You should first reduce the proof to the case $R=1$.

(ii) You may use that for any $A>0$, there exists an even $f_A \in C^k(\mathbb{R})$, $\operatorname{supp}\left(\hat{f}_A\right) \subseteq [-A, A]$, such that

a. $\|f - f_A\|_\infty \le c(k)A^{-k}\left\|f^{(k)}\right\|_\infty$,

b. $|f(t) - f_A(t)| \le c(k)A^{-k}\|f\|_\infty (1+t)^{-k}$, $t \ge 2$.

(iii)You may use that for even f_A, $\operatorname{supp}\left(\hat{f}_A\right) \subseteq [-A, A]$, $\hat{f}_A \in W_1^m$ for $m > d$

$$f_A\left(\delta\sqrt{L}\right)(x, y) = 0, \quad \text{if } \tilde{c}\delta A < \rho(x, y).$$