

A new proof of the atomic decomposition of Hardy spaces

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Abstract. A new proof is given of the atomic decomposition of Hardy spaces H^p , $0 < p \leq 1$, in the classical setting on \mathbb{R}^n . The new method can be used to establish atomic decomposition of maximal Hardy spaces in general and nonclassical settings.

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1. Introduction

The study of the real-variable Hardy spaces H^p , $0 < p \leq 1$, on \mathbb{R}^n was pioneered by Stein and Weiss [6] and a major step forward in developing this theory was made by Fefferman and Stein in [3], see also [5]. Since then there has been a great deal of work done on Hardy spaces. The atomic decomposition of H^p was first established by Coifman [1] in dimension $n = 1$ and by Latter [4] in dimensions $n > 1$.

The purpose of this article is to give a new proof of the atomic decomposition of the H^p spaces in the classical setting on \mathbb{R}^n . Our method does not use the Calderón-Zygmund decomposition of functions and an approximation of the identity as the classical argument does, see [5]. The main advantage of the new proof over the classical one is that it is amenable to utilization in more general and nonclassical settings. For instance, it is used in [2] for establishing the equivalence of maximal and atomic Hardy spaces in the general setting of a metric measure space with the doubling property and in the presence of a non-negative self-adjoint operator whose heat kernel has Gaussian localization.

Notation. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$ and by $B(x, \delta)$ the open ball centered at $x \in \mathbb{R}^n$ of radius δ , i.e. $B(x, \delta) := \{y \in \mathbb{R}^n : |x - y| < \delta\}$. Positive constants will be denoted by c, c_1, \dots and they may vary at every occurrence; $a \sim b$ will stand for $c_1 \leq a/b \leq c_2$.

1.1. Maximal operators and H^p spaces

We begin by recalling some basic facts about Hardy spaces on \mathbb{R}^n . For a complete account of Hardy spaces we refer the reader to [5].

Given $\varphi \in \mathcal{S}$ with \mathcal{S} being the Schwartz class on \mathbb{R}^n and $f \in \mathcal{S}'$ one defines

$$M_\varphi f(x) := \sup_{t>0} |\varphi_t * f(x)| \quad \text{with } \varphi_t(x) := t^{-n} \varphi(t^{-1}x), \quad \text{and} \quad (1)$$

$$M_{\varphi,a}^* f(x) := \sup_{t>0} \sup_{y \in \mathbb{R}^n, |x-y| \leq at} |\varphi_t * f(y)|, \quad a \geq 1. \quad (2)$$

We now recall the grand maximal operator. Write

$$\mathcal{P}_N(\varphi) := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \max_{|\alpha| \leq N+1} |\partial^\alpha \varphi(x)|$$

and denote

$$\mathcal{F}_N := \{\varphi \in \mathcal{S} : \mathcal{P}_N(\varphi) \leq 1\}.$$

The grand maximal operator \mathcal{M}_N is defined by

$$\mathcal{M}_N f(x) := \sup_{\varphi \in \mathcal{F}_N} M_{\varphi,1}^* f(x), \quad f \in \mathcal{S}'. \quad (3)$$

It is easy to see that for any $\varphi \in \mathcal{S}$ and $a \geq 1$ one has

$$M_{\varphi,a}^* f(x) \leq a^N \mathcal{P}_N(\varphi) \mathcal{M}_N f(x), \quad f \in \mathcal{S}'. \quad (4)$$

Definition 1. *The space H^p , $0 < p \leq 1$, is defined as the set of all bounded distributions $f \in \mathcal{S}'$ such that the Poisson maximal function $\sup_{t>0} |P_t * f(x)|$ belongs to L^p ; the quasi-norm on H^p is defined by*

$$\|f\|_{H^p} := \left\| \sup_{t>0} |P_t * f(\cdot)| \right\|_{L^p}. \quad (5)$$

As is well known the following assertion holds, see [3, 5]:

Proposition 1. *Let $0 < p \leq 1$, $a \geq 1$, and assume $\varphi \in \mathcal{S}$ and $\int_{\mathbb{R}^n} \varphi \neq 0$. Then for any $N \geq \lfloor \frac{n}{p} \rfloor + 1$*

$$\|f\|_{H^p} \sim \|M_{\varphi,a}^* f\|_{L^p} \sim \|\mathcal{M}_N f\|_{L^p}, \quad \forall f \in H^p. \quad (6)$$

1.2. Atomic H^p spaces

A function $a \in L^\infty(\mathbb{R}^n)$ is called an atom if there exists a ball B such that

- (i) $\text{supp } a \subset B$,
- (ii) $\|a\|_{L^\infty} \leq |B|^{-1/p}$, and

(iii) $\int_{\mathbb{R}^n} x^\alpha a(x) dx = 0$ for all α with $|\alpha| \leq n(p^{-1} - 1)$.

The atomic Hardy space H_A^p , $0 < p \leq 1$, is defined as the set of all distributions $f \in \mathcal{S}'$ that can be represented in the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty, \quad (7)$$

$\{a_j\}$ are atoms, and the convergence is in \mathcal{S}' . Set

$$\|f\|_{H_A^p} := \inf_{f = \sum_j \lambda_j a_j} \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}, \quad f \in H_A^p. \quad (8)$$

2. Atomic decomposition of H^p spaces

We now come to the main point in this article, that is, to give a new proof of the following classical result [1, 4], see also [5]:

Theorem 1. *For any $0 < p \leq 1$ the continuous embedding $H^p \subset H_A^p$ is valid, that is, if $f \in H^p$, then $f \in H_A^p$ and*

$$\|f\|_{H_A^p} \leq c \|f\|_{H^p}, \quad (9)$$

where $c > 0$ is a constant depending only on p, n . This along with the easy to prove embedding $H_A^p \subset H^p$ leads to $H^p = H_A^p$ and $\|f\|_{H^p} \sim \|f\|_{H_A^p}$ for $f \in H^p$.

Proof. We first derive a simple *decomposition* identity which will play a central rôle in this proof. For this construction we need the following

Lemma 1. *For any $m \geq 1$ there exists a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp } \varphi \subset B(0, 1)$, $\hat{\varphi}(0) = 1$, and $\partial^\alpha \hat{\varphi}(0) = 0$ for $0 < |\alpha| \leq m$. Here $\hat{\varphi}$ is the Fourier transform of φ , defined by $\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-ix \cdot \xi} dx$.*

Proof. We will construct a function φ with the claimed properties in dimension $n = 1$. Then a normalized dilation of $\varphi(x_1)\varphi(x_2) \cdots \varphi(x_n)$ will have the claimed properties on \mathbb{R}^n .

For the univariate construction, pick a smooth ‘‘bump’’ ϕ with the following properties: $\phi \in C_0^\infty(\mathbb{R})$, $\text{supp } \phi \subset [-1/4, 1/4]$, $\phi(x) > 0$ for $x \in (-1/4, 1/4)$, and ϕ is even. Let $\Theta(x) := \phi(x + 1/2) - \phi(x - 1/2)$ for $x \in \mathbb{R}$. Clearly Θ is odd.

We may assume that $m \geq 1$ is even, otherwise we work with $m + 1$ instead. Denote $\Delta_h^m := (T_h - T_{-h})^m$, where $T_h f(x) := f(x + h)$.

We define $\varphi(x) := \frac{1}{8^m} \Delta_h^m \Theta(x)$, where $h = \frac{1}{8^m}$. Clearly, $\varphi \in C_0^\infty(\mathbb{R})$, φ is even, and $\text{supp } \varphi \subset [-\frac{7}{8}, -\frac{1}{8}] \cup [\frac{1}{8}, \frac{7}{8}]$. It is readily seen that for $\nu = 1, 2, \dots, m$

$$\hat{\varphi}^{(\nu)}(\xi) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1} \Delta_h^m \Theta(x) e^{-i\xi x} dx$$

and hence

$$\hat{\varphi}^{(\nu)}(0) = (-i)^\nu \int_{\mathbb{R}} x^{\nu-1} \Delta_h^m \Theta(x) dx = (-i)^{\nu+m} \int_{\mathbb{R}} \Theta(x) \Delta_h^m x^{\nu-1} dx = 0.$$

On the other hand,

$$\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) dx = 2 \int_0^\infty x^{-1} \Delta_h^m \Theta(x) dx = 2(-1)^m \int_{1/4}^{3/4} \Theta(x) \Delta_h^m x^{-1} dx.$$

However, if f a sufficiently smooth function, then $\Delta_h^m f(x) = (2h)^m f^{(m)}(\xi)$, where $\xi \in (x - mh, x + mh)$. Hence,

$$\Delta_h^m x^{-1} = (2h)^m m! (-1)^m \xi^{-m-1} \quad \text{with } \xi \in (x - mh, x + mh) \subset [1/8, 7/8].$$

Consequently, $\hat{\varphi}(0) \neq 0$ and then $\hat{\varphi}(0)^{-1} \varphi(x)$ has the claimed properties. \square

With the aid of the above lemma, we pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ with the following properties: $\text{supp } \varphi \subset B(0, 1)$, $\hat{\varphi}(0) = 1$, and $\partial^\alpha \hat{\varphi}(0) = 0$ for $0 < |\alpha| \leq K$, where K is sufficiently large. More precisely, we choose $K \geq n/p$.

Set $\psi(x) := 2^n \varphi(2x) - \varphi(x)$. Then $\hat{\psi}(\xi) = \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi)$. Therefore, $\partial^\alpha \hat{\psi}(0) = 0$ for $|\alpha| \leq K$ which implies $\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$ for $|\alpha| \leq K$. We also introduce the function $\tilde{\psi}(x) := 2^n \varphi(2x) + \varphi(x)$. We will use the notation $h_k(x) := 2^{kn} h(2^k x)$.

Clearly, for any $f \in \mathcal{S}'$ we have $f = \lim_{j \rightarrow \infty} \varphi_j * \varphi_j * f$ (convergence in \mathcal{S}'), which leads to the following representation: For any $j \in \mathbb{Z}$

$$\begin{aligned} f &= \varphi_j * \varphi_j * f + \sum_{k=j}^{\infty} [\varphi_{k+1} * \varphi_{k+1} * f - \varphi_k * \varphi_k * f] \\ &= \varphi_j * \varphi_j * f + \sum_{k=j}^{\infty} [\varphi_{k+1} - \varphi_k] * [\varphi_{k+1} + \varphi_k] * f. \end{aligned}$$

Thus we arrive at

$$f = \varphi_j * \varphi_j * f + \sum_{k=j}^{\infty} \psi_k * \tilde{\psi}_k * f, \quad \forall f \in \mathcal{S}' \quad \forall j \in \mathbb{Z} \quad (\text{convergence in } \mathcal{S}'). \quad (10)$$

Observe that $\text{supp } \psi_k \subset B(0, 2^{-k})$ and $\text{supp } \tilde{\psi}_k \subset B(0, 2^{-k})$.

In what follows we will utilize the grand maximal operator \mathcal{M}_N , defined in (3) with $N := \lfloor \frac{n}{p} \rfloor + 1$. The following claim follows readily from (4): If $\phi \in \mathcal{S}$, then for any $f \in \mathcal{S}'$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$

$$|\phi_k * f(y)| \leq c \mathcal{M}_N f(x) \quad \text{for all } y \in \mathbb{R}^n \text{ with } |y - x| \leq 2^{-k+1}, \quad (11)$$

where the constant $c > 0$ depends only on $\mathcal{P}_N(\phi)$ and N .

Let $f \in H^p$, $0 < p \leq 1$, $f \neq 0$. We define

$$\Omega_r := \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^r\}, \quad r \in \mathbb{Z}. \quad (12)$$

Clearly, Ω_r is open, $\Omega_{r+1} \subset \Omega_r$, and $\mathbb{R}^n = \cup_{r \in \mathbb{Z}} \Omega_r$. It is easy to see that

$$\sum_{r \in \mathbb{Z}} 2^{pr} |\Omega_r| \leq c \int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p dx \leq c \|f\|_{H^p}^p. \quad (13)$$

Indeed, we have

$$\begin{aligned} \sum_{r \in \mathbb{Z}} 2^{pr} |\Omega_r| &= \sum_{r \in \mathbb{Z}} 2^{pr} \sum_{\nu \geq r} |\Omega_\nu \setminus \Omega_{\nu+1}| = \sum_{\nu \in \mathbb{Z}} |\Omega_\nu \setminus \Omega_{\nu+1}| \sum_{r \leq \nu} 2^{pr} \\ &\leq c_p \sum_{\nu \in \mathbb{Z}} 2^{p\nu} |\Omega_\nu \setminus \Omega_{\nu+1}| \leq c_p \sum_{\nu \in \mathbb{Z}} \int_{\Omega_\nu \setminus \Omega_{\nu+1}} \mathcal{M}_N f(x)^p dx \\ &= c_p \int_{\mathbb{R}^n} \mathcal{M}_N f(x)^p dx. \end{aligned}$$

From (13) we get $|\Omega_r| \leq c 2^{-pr} \|f\|_{H^p}^p$ for $r \in \mathbb{Z}$. Therefore, for any $r \in \mathbb{Z}$ there exists $J > 0$ such that $\|\varphi_j * \varphi_j * f\|_\infty \leq c 2^r$ for $j < -J$. Consequently, $\|\varphi_j * \varphi_j * f\|_\infty \rightarrow 0$ as $j \rightarrow -\infty$, which implies

$$f = \lim_{K \rightarrow \infty} \sum_{k=-\infty}^K \psi_k * \tilde{\psi}_k * f \quad (\text{convergence in } \mathcal{S}'). \quad (14)$$

Assuming that $\Omega_r \neq \emptyset$ we write

$$E_{rk} := \{x \in \Omega_r : \text{dist}(x, \Omega_r^c) > 2^{-k+1}\} \setminus \{x \in \Omega_{r+1} : \text{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\}.$$

By (13) it follows that $|\Omega_r| < \infty$ and hence there exists $s_r \in \mathbb{Z}$ such that $E_{rs_r} \neq \emptyset$ and $E_{rk} = \emptyset$ for $k < s_r$. Evidently $s_r \leq s_{r+1}$. We define

$$F_r(x) := \sum_{k \geq s_r} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy, \quad x \in \mathbb{R}^n, \quad r \in \mathbb{Z}, \quad (15)$$

and more generally

$$F_{r, \kappa_0, \kappa_1}(x) := \sum_{k=\kappa_0}^{\kappa_1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy, \quad s_r \leq \kappa_0 \leq \kappa_1 \leq \infty. \quad (16)$$

It will be shown in Lemma 2 below that the functions F_r and $F_{r, \kappa_0, \kappa_1}$ are well defined and $F_r, F_{r, \kappa_0, \kappa_1} \in L^\infty$.

Note that $\text{supp } \psi_k \subset B(0, 2^{-k})$ and hence

$$\text{supp} \left(\int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \right) \subset \{x : \text{dist}(x, E_{rk}) < 2^{-k}\}. \quad (17)$$

On the other hand, clearly $B(y, 2^{-k+1}) \cap (\Omega_r \setminus \Omega_{r+1}) \neq \emptyset$ for each $y \in E_{rk}$, and $\mathcal{P}_N(\tilde{\psi}) \leq c$. Therefore, see (11), $|\tilde{\psi}_k * f(y)| \leq c2^r$ for $y \in E_{rk}$, which implies

$$\left\| \int_E \psi_k(\cdot - y) \tilde{\psi}_k * f(y) dy \right\|_{\infty} \leq c2^r, \quad \forall E \subset E_{rk}. \quad (18)$$

Similarly,

$$\left\| \int_E \varphi_k(\cdot - y) \tilde{\varphi}_k * f(y) dy \right\|_{\infty} \leq c2^r, \quad \forall E \subset E_{rk}. \quad (19)$$

We collect all we need about the functions F_r and $F_{r, \kappa_0, \kappa_1}$ in the following

Lemma 2. (a) *We have*

$$E_{rk} \cap E_{r'k} = \emptyset \quad \text{if } r \neq r' \quad \text{and} \quad \mathbb{R}^n = \cup_{r \in \mathbb{Z}} E_{rk}, \quad \forall k \in \mathbb{Z}. \quad (20)$$

(b) *There exists a constant $c > 0$ such that for any $r \in \mathbb{Z}$ and $s_r \leq \kappa_0 \leq \kappa_1 \leq \infty$*

$$\|F_r\|_{\infty} \leq c2^r, \quad \|F_{r, \kappa_0, \kappa_1}\|_{\infty} \leq c2^r. \quad (21)$$

(c) *The series in (15) and (16) (if $\kappa_1 = \infty$) converge point-wise and in distributional sense.*

(d) *Moreover,*

$$F_r(x) = 0, \quad \forall x \in \mathbb{R}^n \setminus \Omega_r, \quad \forall r \in \mathbb{Z}. \quad (22)$$

Proof. Identities (20) are obvious and (22) follows readily from (17).

We next prove the left-hand side inequality in (21); the proof of the right-hand side inequality is similar and will be omitted. Consider the case when $\Omega_{r+1} \neq \emptyset$ (the case when $\Omega_{r+1} = \emptyset$ is easier). Write

$$\begin{aligned} U_k &:= \{x \in \Omega_r : \text{dist}(x, \Omega_r^c) > 2^{-k+1}\}, \\ V_k &:= \{x \in \Omega_{r+1} : \text{dist}(x, \Omega_{r+1}^c) > 2^{-k+1}\}. \end{aligned}$$

Observe that $E_{rk} = U_k \setminus V_k$.

Clearly, (17) implies $|F_r(x)| = 0$ for $x \in \mathbb{R}^n \setminus \cup_{k \geq s_r} \{x : \text{dist}(x, E_{rk}) < 2^{-k}\}$. We next estimate $|F_r(x)|$ for $x \in \cup_{k \geq s_r} \{x : \text{dist}(x, E_{rk}) < 2^{-k}\}$. Two cases present themselves here.

Case 1: $x \in [\cup_{k \geq s_r} \{x : \text{dist}(x, E_{rk}) < 2^{-k}\}] \cap \Omega_{r+1}$. Then there exist $\nu, \ell \in \mathbb{Z}$ such that

$$x \in (U_{\ell+1} \setminus U_{\ell}) \cap (V_{\nu+1} \setminus V_{\nu}). \quad (23)$$

Due to $\Omega_{r+1} \subset \Omega_r$ we have $V_k \subset U_k$, implying $(U_{\ell+1} \setminus U_{\ell}) \cap (V_{\nu+1} \setminus V_{\nu}) = \emptyset$ if $\nu < \ell$. We next consider two subcases depending on whether $\nu \geq \ell + 3$ or $\ell \leq \nu \leq \ell + 2$.

(a) Let $\nu \geq \ell + 3$. We claim that (23) yields

$$B(x, 2^{-k}) \cap E_{rk} = \emptyset \quad \text{for } k \geq \nu + 2 \quad \text{or } k \leq \ell - 1. \quad (24)$$

Indeed, if $k \geq \nu + 2$, then $E_{rk} \subset \Omega_r \setminus V_{\nu+2}$, which implies (24), while if $k \leq \ell - 1$, then $E_{rk} \subset U_{\ell-1}$, again implying (24).

We also claim that

$$B(x, 2^{-k}) \subset E_{rk} \quad \text{for } \ell + 2 \leq k \leq \nu - 1. \quad (25)$$

Indeed, clearly

$$(U_{\ell+1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_\nu) \subset (U_{k-1} \setminus U_\ell) \cap (V_{\nu+1} \setminus V_{\nu+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies (25).

From (17) and (24)- (25) it follows that

$$\begin{aligned} F_r(x) &= \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy = \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \\ &\quad + \sum_{k=\ell+2}^{\nu-2} \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y) dy + \sum_{k=\nu-1}^{\nu+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy. \end{aligned}$$

However,

$$\begin{aligned} &\sum_{k=\ell+2}^{\nu-2} \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \\ &= \sum_{k=\ell+2}^{\nu-2} [\varphi_{k+1} * \varphi_{k+1} * f(x) - \varphi_k * \varphi_k * f(x)] \\ &= \varphi_{\nu-1} * \varphi_{\nu-1} * f(x) - \varphi_{\ell+2} * \varphi_{\ell+2} * f(x) \\ &= \int_{E_{r,\nu-1}} \varphi_{\nu-1}(x-y) \varphi_{\nu-1} * f(y) dy - \int_{E_{r,\ell+2}} \varphi_{\ell+2}(x-y) \varphi_{\ell+2} * f(y) dy. \end{aligned}$$

Combining the above with (18) and (19) we obtain $|F_r(x)| \leq c2^r$.

(b) Let $\ell \leq \nu \leq \ell + 2$. Just as above we have

$$F_r(x) = \sum_{k=\ell}^{\nu+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy = \sum_{k=\ell}^{\ell+3} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy.$$

We use (18) to estimate each the above four integrals and obtain $|F_r(x)| \leq c2^r$.

Case 2: $x \in \Omega_r \setminus \Omega_{r+1}$. Then there exists $\ell \geq s_r$ such that

$$x \in (U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}).$$

Just as in the proof of (24) we have $B(x, 2^{-k}) \cap E_{rk} = \emptyset$ for $k \leq \ell - 1$, and as in the proof of (25) we have

$$(U_{\ell+1} \setminus U_\ell) \cap (\Omega_r \setminus \Omega_{r+1}) \subset U_{k-1} \setminus V_{k+1},$$

which implies $B(x, 2^{-k}) \subset E_{rk}$ for $k \geq \ell + 2$. We use these and (17) to obtain

$$\begin{aligned} F_r(x) &= \sum_{k=\ell}^{\infty} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \\ &= \sum_{k=\ell}^{\ell+1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy + \sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y) dy. \end{aligned}$$

For the last sum we have

$$\begin{aligned} \sum_{k=\ell+2}^{\infty} \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y) dy &= \lim_{\nu \rightarrow \infty} \sum_{k=\ell+2}^{\nu} \psi_k * \tilde{\psi}_k * f(x) \\ &= \lim_{\nu \rightarrow \infty} (\varphi_{\nu+1} * \varphi_{\nu+1} * f(x) - \varphi_{\ell+2} * \varphi_{\ell+2} * f(x)) \\ &= \lim_{\nu \rightarrow \infty} \left(\int_{E_{r, \nu+1}} \varphi_{\nu+1}(x-y) \varphi_{\nu+1} * f(y) dy \right. \\ &\quad \left. - \int_{E_{r, \ell+2}} \varphi_{\ell+2}(x-y) \varphi_{\ell+2} * f(y) dy \right). \end{aligned}$$

From the above and (18)-(19) we obtain $|F_r(x)| \leq c2^r$.

The point-wise convergence of the series in (15) follows from above and we similarly establish the point-wise convergence in (16).

The convergence in distributional sense in (15) relies on the following assertion: For every $\phi \in \mathcal{S}$

$$\sum_{k \geq s_r} |\langle g_{rk}, \phi \rangle| < \infty, \quad \text{where } g_{rk}(x) := \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy. \quad (26)$$

Here $\langle g_{rk}, \phi \rangle := \int_{\mathbb{R}^n} g_{rk} \bar{\phi} dx$. To prove the above we will employ this estimate:

$$\|\tilde{\psi}_k f\|_{\infty} \leq c2^{kn/p} \|f\|_{H^p}, \quad k \in \mathbb{Z}. \quad (27)$$

Indeed, using (4) we get

$$\begin{aligned} |\tilde{\psi}_k f(x)|^p &\leq \inf_{y: |x-y| \leq 2^{-k}} \sup_{z: |y-z| \leq 2^{-k}} |\tilde{\psi}_k f(z)|^p \leq \inf_{y: |x-y| \leq 2^{-k}} c \mathcal{M}_N(f)(y)^p \\ &\leq c |B(x, 2^{-k})|^{-1} \int_{B(x, 2^{-k})} \mathcal{M}_N(f)(y)^p dy \leq c2^{kn} \|f\|_{H^p}^p, \end{aligned}$$

and (27) follows.

We will also need the following estimate: For any $\sigma > n$ there exists a constant $c_{\sigma} > 0$ such that

$$\left| \int_{\mathbb{R}^n} \psi_k(x-y) \phi(x) dx \right| \leq c_{\sigma} 2^{-k(K+1)} (1 + |y|)^{-\sigma}, \quad y \in \mathbb{R}^n, k \geq 0. \quad (28)$$

This is a standard estimate for inner products taking into account that $\phi \in \mathcal{S}$ and $\psi \in C^\infty$, $\text{supp } \psi \subset B(0, 1)$, and $\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$ for $|\alpha| \leq K$.

We now estimate $|\langle g_{rk}, \phi \rangle|$. From (27) and the fact that $\psi \in C_0^\infty(\mathbb{R})$ and $\phi \in \mathcal{S}$ it readily follows that

$$\int_{E_{rk}} \int_{\mathbb{R}^n} |\psi_k(x-y)| |\phi(x)| |\tilde{\psi}_k f(y)| dy dx < \infty, \quad k \geq s_r.$$

Therefore, we can use Fubini's theorem, (27), and (28) to obtain for $k \geq 0$

$$\begin{aligned} |\langle g_{rk}, \phi \rangle| &\leq \int_{E_{rk}} \left| \int_{\mathbb{R}^n} \psi_k(x-y) \phi(x) dx \right| |\tilde{\psi}_k f(y)| dy \\ &\leq c 2^{-k(K+1-n/p)} \|f\|_{H^p} \int_{E_{rk}} (1+|y|)^{-\sigma} dy \leq c 2^{-k(K+1-n/p)} \|f\|_{H^p}, \end{aligned} \quad (29)$$

which implies (26) because $K \geq n/p$.

Denote $G_\ell := \sum_{k=s_r}^\ell g_{rk}$. From the above proof of (b) and (21) we infer that $G_\ell(x) \rightarrow F_r(x)$ as $\ell \rightarrow \infty$ for $x \in \mathbb{R}^n$ and $\|G_\ell\|_\infty \leq c 2^r < \infty$ for $\ell \geq s_r$. On the other hand, from (26) it follows that the series $\sum_{k \geq s_r} g_{rk}$ converges in distributional sense. By applying the dominated convergence theorem one easily concludes that $F_r = \sum_{k \geq s_r} g_{rk}$ with the convergence in distributional sense. \square

We set $F_r := 0$ in the case when $\Omega_r = \emptyset$, $r \in \mathbb{Z}$.

Note that by (20) it follows that

$$\psi_k * \tilde{\psi}_k * f(x) = \int_{\mathbb{R}^n} \psi_k(x-y) \tilde{\psi}_k * f(y) dy = \sum_{r \in \mathbb{Z}} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \quad (30)$$

and using (14) and the definition of F_r in (15) we arrive at

$$f = \sum_{r \in \mathbb{Z}} F_r \text{ in } \mathcal{S}', \text{ i.e. } \langle f, \phi \rangle = \sum_{r \in \mathbb{Z}} \langle F_r, \phi \rangle, \quad \forall \phi \in \mathcal{S}, \quad (31)$$

where the last series converges absolutely. Above $\langle f, \phi \rangle$ denotes the action of f on $\bar{\phi}$. We next provide the needed justification of equality (31).

From (14), (15), (30), and the notation from (26) we obtain for $\phi \in \mathcal{S}$

$$\langle f, \phi \rangle = \sum_k \langle \psi_k * \tilde{\psi}_k * f, \phi \rangle = \sum_k \sum_r \langle g_{rk}, \phi \rangle = \sum_r \sum_k \langle g_{rk}, \phi \rangle = \sum_r \langle F_r, \phi \rangle.$$

Clearly, to justify the above it suffices to show that $\sum_k \sum_r |\langle g_{rk}, \phi \rangle| < \infty$. We split this sum into two:

$$\sum_k \sum_r |\langle g_{rk}, \phi \rangle| = \sum_{k \geq 0} \sum_r |\langle g_{rk}, \phi \rangle| + \sum_{k < 0} \sum_r |\langle g_{rk}, \phi \rangle| =: \Sigma_1 + \Sigma_2.$$

To estimate Σ_1 we use (29) and obtain

$$\begin{aligned}\Sigma_1 &\leq c\|f\|_{H^p} \sum_{k \geq 0} 2^{-k(K+1-n/p)} \sum_r \int_{E_{rk}} (1+|y|)^{-\sigma} dy \\ &\leq c\|f\|_{H^p} \sum_{k \geq 0} 2^{-k(K+1-n/p)} \int_{\mathbb{R}^n} (1+|y|)^{-\sigma} dy \leq c\|f\|_{H^p}.\end{aligned}$$

Here we also used that $K \geq n/p$ and $\sigma > n$.

We estimate Σ_2 in a similar manner, using that $\int_{\mathbb{R}^n} |\psi_k(y)| dy \leq c < \infty$ and (27). We get

$$\begin{aligned}\Sigma_2 &\leq c\|f\|_{H^p} \sum_{k < 0} 2^{kn/p} \sum_r \int_{E_{rk}} \int_{\mathbb{R}^n} |\psi_k(x-y)| dy |\phi(x)| dx \\ &\leq c\|f\|_{H^p} \sum_{k < 0} 2^{kn/p} \int_{\mathbb{R}^n} (1+|x|)^{-n-1} dx \leq c\|f\|_{H^p}.\end{aligned}$$

The above estimates of Σ_1 and Σ_2 imply $\sum_k \sum_r |\langle g_{rk}, \phi \rangle| < \infty$, which completes the justification of (31).

Observe that due to $\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$ for $|\alpha| \leq K$ we have

$$\int_{\mathbb{R}^n} x^\alpha F_r(x) dx = 0 \quad \text{for } |\alpha| \leq K, r \in \mathbb{Z}. \quad (32)$$

We next decompose each function F_r into atoms. To this end we need a Whitney type cover for Ω_r , given in the following

Lemma 3. *Suppose Ω is an open proper subset of \mathbb{R}^n and let $\rho(x) := \text{dist}(x, \Omega^c)$. Then there exists a constant $K > 0$, depending only on n , and a sequence of points $\{\xi_j\}_{j \in \mathbb{N}}$ in Ω with the following properties, where $\rho_j := \text{dist}(\xi_j, \Omega^c)$:*

- (a) $\Omega = \cup_{j \in \mathbb{N}} B(\xi_j, \rho_j/2)$.
- (b) $\{B(\xi_j, \rho_j/5)\}$ are disjoint.
- (c) If $B(\xi_j, \frac{3\rho_j}{4}) \cap B(\xi_\nu, \frac{3\rho_\nu}{4}) \neq \emptyset$, then $7^{-1}\rho_\nu \leq \rho_j \leq 7\rho_\nu$.
- (d) For every $j \in \mathbb{N}$ there are at most K balls $B(\xi_\nu, \frac{3\rho_\nu}{4})$ intersecting $B(\xi_j, \frac{3\rho_j}{4})$.

Variants of this simple lemma are well known and frequently used. To prove it one simply selects $\{B(\xi_j, \rho(\xi_j)/5)\}_{j \in \mathbb{N}}$ to be a maximal disjoint subset of $\{B(x, \rho(x)/5)\}_{x \in \Omega}$ and then properties (a)-(d) follow readily, see [5], pp. 15-16.

We apply Lemma 3 to each set $\Omega_r \neq \emptyset$, $r \in \mathbb{Z}$. Fix $r \in \mathbb{Z}$ and assume $\Omega_r \neq \emptyset$. Denote by $B_j := B(\xi_j, \rho_j/2)$, $j = 1, 2, \dots$, the balls given by Lemma 3, applied

to Ω_r , with the additional assumption that these balls are ordered so that $\rho_1 \geq \rho_2 \geq \dots$. We will adhere to the notation from Lemma 3. We will also use the more compact notation $\mathcal{B}_r := \{B_j\}_{j \in \mathbb{N}}$ for the set of balls covering Ω_r .

For each ball $B \in \mathcal{B}_r$ and $k \geq s_r$ we define

$$E_{rk}^B := E_{rk} \cap \{x : \text{dist}(x, B) < 2^{-k+1}\} \quad \text{if } B \cap E_{rk} \neq \emptyset \quad (33)$$

and set $E_{rk}^B := \emptyset$ if $B \cap E_{rk} = \emptyset$.

We also define, for $\ell = 1, 2, \dots$,

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \bigcup_{\nu > \ell} E_{rk}^{B_\nu} \quad \text{and} \quad (34)$$

$$F_{B_\ell}(x) := \sum_{k \geq s_r} \int_{R_{rk}^{B_\ell}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy. \quad (35)$$

Lemma 4. *For every $\ell \geq 1$ the function F_{B_ℓ} is well defined, more precisely, the series in (35) converges point-wise and in distributional sense. Furthermore,*

$$\text{supp } F_{B_\ell} \subset 7B_\ell, \quad (36)$$

$$\int_{\mathbb{R}^n} x^\alpha F_{B_\ell}(x) dx = 0 \quad \text{for all } \alpha \text{ with } |\alpha| \leq n(p^{-1} - 1), \quad (37)$$

and

$$\|F_{B_\ell}\|_\infty \leq c_\sharp 2^r, \quad (38)$$

where the constant c_\sharp is independent of r, ℓ .

In addition, for any $k \geq s_r$

$$E_{rk} = \bigcup_{\ell \geq 1} R_{rk}^{B_\ell} \quad \text{and} \quad R_{rk}^{B_\ell} \cap R_{rk}^{B_m} = \emptyset, \quad \ell \neq m. \quad (39)$$

Hence

$$F_r = \sum_{B \in \mathcal{B}_r} F_B \quad (\text{convergence in } \mathcal{S}'). \quad (40)$$

Proof. Fix $\ell \geq 1$. Observe that using Lemma 3 we have

$$B_\ell \subset \{x \in \mathbb{R}^n : \text{dist}(x, \Omega_r^c) < 2\rho_\ell\}$$

and hence $E_{rk}^{B_\ell} := \emptyset$ if $2^{-k+1} \geq 2\rho_\ell$. Define $k_0 := \min\{k : 2^{-k} < \rho_\ell\}$. Hence $\rho_\ell/2 \leq 2^{-k_0} < \rho_\ell$. Consequently,

$$F_{B_\ell}(x) := \sum_{k \geq k_0} \int_{R_{rk}^{B_\ell}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy. \quad (41)$$

It follows that $\text{supp } F_{B_\ell} \subset B(\xi_\ell, (7/2)\rho_\ell) = 7B_\ell$, which confirms (36).

To prove (38) we will use the following

Lemma 5. For an arbitrary set $S \subset \mathbb{R}^n$ let

$$S_k := \{x \in \mathbb{R}^n : \text{dist}(x, S) < 2^{-k+1}\}$$

and set

$$F_S(x) := \sum_{k \geq \kappa_0} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \quad (42)$$

for some $\kappa_0 \geq s_r$. Then $\|F_S\|_\infty \leq c2^r$, where $c > 0$ is a constant independent of S and κ_0 . Moreover, the above series converges in \mathcal{S}' .

Proof. From (17) it follows that $F_S(x) = 0$ if $\text{dist}(x, S) \geq 3 \times 2^{-\kappa_0}$. Let $x \in S$. Evidently, $B(x, 2^{-k}) \subset S_k$ for every k and hence

$$\begin{aligned} F_S(x) &= \sum_{k \geq \kappa_0} \int_{E_{rk} \cap B(x, 2^{-k})} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \\ &= \sum_{k \geq \kappa_0} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy = F_{r, \kappa_0}(x). \end{aligned}$$

On account of Lemma 2 (b) we obtain $|F_S(x)| = |F_{r, \kappa_0}(x)| \leq c2^r$.

Consider the case when $x \in S_\ell \setminus S_{\ell+1}$ for some $\ell \geq \kappa_0$. Then $B(x, 2^{-k}) \subset S_k$ if $\kappa_0 \leq k \leq \ell - 1$ and $B(x, 2^{-k}) \cap S_k = \emptyset$ if $k \geq \ell + 2$. Therefore,

$$\begin{aligned} F_S(x) &= \sum_{k=\kappa_0}^{\ell-1} \int_{E_{rk}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy + \sum_{k=\ell}^{\ell+1} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y) dy \\ &= F_{r, \kappa_0, \ell-1}(x) + \sum_{k=\ell}^{\ell+1} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y) dy, \end{aligned}$$

where we used the notation from (16). By Lemma 2 (b) and (18) it follows that $|F_S(x)| \leq c2^r$.

We finally consider the case when $2^{-\kappa_0+1} \leq \text{dist}(x, S) < 3 \times 2^{-\kappa_0}$. Then we have $F_S(x) = \int_{E_{r\kappa_0} \cap S_{\kappa_0}} \psi_{\kappa_0}(x-y) \tilde{\psi}_{\kappa_0} * f(y) dy$ and the estimate $|F_S(x)| \leq c2^r$ is immediate from (18).

The convergence in \mathcal{S}' in (42) is established as in the proof of Lemma 2. \square

Fix $\ell \geq 1$ and let $\{B_j : j \in \mathcal{J}\}$ be the set of all balls $B_j = B(\xi_j, \rho_j/2)$ such that $j > \ell$ and

$$B\left(\xi_j, \frac{3\rho_j}{4}\right) \cap B\left(\xi_\ell, \frac{3\rho_\ell}{4}\right) \neq \emptyset.$$

By Lemma 3 it follows that $\#\mathcal{J} \leq K$ and $7^{-1}\rho_\ell \leq \rho_j \leq 7\rho_\ell$ for $j \in \mathcal{J}$. Define

$$k_1 := \min \left\{ k : 2^{-k+1} < 4^{-1} \min \{ \rho_j : j \in \mathcal{J} \cup \{\ell\} \} \right\}. \quad (43)$$

From this definition and $2^{-k_0} < \rho_\ell$ we infer

$$2^{-k_1+1} \geq 8^{-1} \min \{ \rho_j : j \in \mathcal{J} \cup \{\ell\} \} > 8^{-2}\rho_\ell > 8^{-2}2^{-k_0} \implies k_1 \leq k_0 + 7. \quad (44)$$

Clearly, from (43)

$$\{x : \text{dist}(x, B_j) < 2^{-k+1}\} \subset B(\xi_j, 3\rho_j/4), \quad \forall k \geq k_1, \quad \forall j \in \mathcal{J} \cup \{\ell\}. \quad (45)$$

Denote $S := \cup_{j \in \mathcal{J}} B_j$ and $\tilde{S} := \cup_{j \in \mathcal{J}} B_j \cup B_\ell = S \cup B_\ell$. As in Lemma 5 we set

$$S_k := \{x \in \mathbb{R}^n : \text{dist}(x, S) < 2^{-k+1}\} \text{ and } \tilde{S}_k := \{x \in \mathbb{R}^n : \text{dist}(x, \tilde{S}) < 2^{-k+1}\}.$$

It readily follows from the definition of k_1 in (43) that

$$R_{rk}^{B_\ell} := E_{rk}^{B_\ell} \setminus \cup_{\nu > \ell} E_{rk}^{B_\nu} = (E_{rk} \cap \tilde{S}_k) \setminus (E_{rk} \cap S_k) \quad \text{for } k \geq k_1. \quad (46)$$

Denote

$$F_S(x) := \sum_{k \geq k_1} \int_{E_{rk} \cap S_k} \psi_k(x-y) \tilde{\psi}_k * f(y) dy, \quad \text{and}$$

$$F_{\tilde{S}}(x) := \sum_{k \geq k_1} \int_{E_{rk} \cap \tilde{S}_k} \psi_k(x-y) \tilde{\psi}_k * f(y) dy.$$

From (46) and the fact that $S \subset \tilde{S}$ it follows that

$$F_{B_\ell}(x) = F_{\tilde{S}}(x) - F_S(x) + \sum_{k_0 \leq k < k_1} \int_{R_{rk}^{B_\ell}} \psi_k(x-y) \tilde{\psi}_k * f(y) dy.$$

By Lemma 5 we get $\|F_S\|_\infty \leq c2^r$ and $\|F_{\tilde{S}}\|_\infty \leq c2^r$. On the other hand from (44) we have $k_1 - k_0 \leq 7$. We estimate each of the (at most 7) integrals above using (18) to conclude that $\|F_{B_\ell}\|_\infty \leq c2^r$.

We deal with the convergence in (35) and (40) as in the proof of Lemma 2.

Clearly, (37) follows from the fact that $\int_{\mathbb{R}^n} x^\alpha \psi(x) dx = 0$ for all α with $|\alpha| \leq K$.

Finally, from Lemma 3 we have $\Omega_r \subset \cup_{j \in \mathbb{N}} B_\ell$ and then (39) is immediate from (33) and (34). \square

We are now prepared to complete the proof of Theorem 1. For every ball $B \in \mathcal{B}_r$, $r \in \mathbb{Z}$, provided $\Omega_r \neq \emptyset$, we define $B^* := 7B$,

$$a_B(x) := c_\sharp^{-1} |B^*|^{-1/p} 2^{-r} F_B(x) \quad \text{and} \quad \lambda_B := c_\sharp |B^*|^{1/p} 2^r,$$

where $c_\sharp > 0$ is the constant from (38). By (36) $\text{supp } a_B \subset B^*$ and by (38)

$$\|a_B\|_\infty \leq c_\sharp^{-1} |B^*|^{-1/p} 2^{-r} \|F_B\|_\infty \leq |B^*|^{-1/p}.$$

Furthermore, from (37) it follows that $\int_{\mathbb{R}^n} x^\alpha a_B(x) dx = 0$ if $|\alpha| \leq n(p^{-1} - 1)$. Therefore, each a_B is an atom for H^p .

We set $\mathcal{B}_r := \emptyset$ if $\Omega_r = \emptyset$. Now, using the above, (31), and Lemma 4 we get

$$f = \sum_{r \in \mathbb{Z}} F_r = \sum_{r \in \mathbb{Z}} \sum_{B \in \mathcal{B}_r} F_B = \sum_{r \in \mathbb{Z}} \sum_{B \in \mathcal{B}_r} \lambda_B a_B,$$

where the convergence is in \mathcal{S}' , and

$$\sum_{r \in \mathbb{Z}} \sum_{B \in \mathcal{B}_r} |\lambda_B|^p \leq c \sum_{r \in \mathbb{Z}} 2^{pr} \sum_{B \in \mathcal{B}_r} |B| = c \sum_{r \in \mathbb{Z}} 2^{pr} |\Omega_r| \leq c \|f\|_{H^p}^p,$$

which is the claimed atomic decomposition of $f \in H^p$. Above we used that $|B^*| = |7B| = 7^n |B|$. The proof of Theorem 1 is complete. \square

Remark. The proof of Theorem 1 can be considerably simplified and shortened if one seeks to establish atomic decomposition of the H^p spaces in terms of q -atoms with $p < q < \infty$ rather than ∞ -atoms as in Theorem 1, i.e. atoms satisfying $\|a\|_{L^q} \leq |B|^{1/q-1/p}$ with $q < \infty$ rather than $\|a\|_{L^\infty} \leq |B|^{-1/p}$. We will not elaborate on this here.

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