Mathematical foundations of Machine Learning 2023 – lesson 2

Shai Dekel

TEL AVIV UNIVERSITY

Banach Spaces

Definition Banach space is a complete normed vector space B over a field $F = \{\mathbb{R}, \mathbb{C}\}$,

Vector space: $\exists 0 \in B$, $\forall f, g \in B$, $\alpha, \beta \in F \Rightarrow \alpha f + \beta g \in B$.
Complete: Every Cauchy sequence in B converges to an element of B.
Norm:

i. $f \neq 0 \Rightarrow \|f\| > 0$

ii. $\|\alpha f\| = |\alpha| \|f\|$, $\forall \alpha \in F$,

iii. Triangle inequality $\|f + g\| \leq \|f\| + \|g\|$

Measure

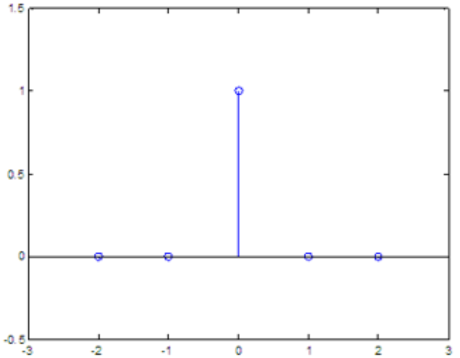
In this course we shall mostly use the standard Lebesgue measure – the volume of a (measurable) set.

Examples: $\Omega = [0,2]^n \subset \mathbb{R}^n$, $\mu(\Omega) = |\Omega| = 2^n$.

We will need the notion of zero measure (volume). Example: a set of discrete points.

Lp Spaces

$\Omega \subseteq \mathbb{R}^n$ domain. Examples: $\Omega = [a,b] \subset \mathbb{R}$, $\Omega = [0,1]^n \subset \mathbb{R}^n, \Omega = \mathbb{R}^n$.



$$\|f\|_{L_p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|, & p = \infty. \end{cases}$$

$$\operatorname*{ess\,sup}_{x} |f(x)| := \sup_{A>0} \left\{ A > 0 : \ \left|\{x : |f(x)| \geq A\}\right| > 0 \right\}.$$

$1 \leq p \leq \infty$ Banach spaces

$0 < p < 1$ Quasi-Banach spaces (quasi-triangle inequality holds)

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p .$$

Theorem [Hölder] $1 \leq p \leq \infty$, $f \in L_p(\Omega), g \in L_{p'}(\Omega)$

$$\left|\int_\Omega fg\right| \leq \int_\Omega |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1 .$$

Lemma Young's inequality for products,

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \ \forall a, b \ge 0.$$

Proof of lemma The logarithmic function is concave. Therefore

$$\begin{aligned}\log\left(\frac{1}{p}a^p + \frac{1}{p'}b^{p'}\right) &= \log\left(\frac{1}{p}a^p + \left(1 - \frac{1}{p}\right)b^{p'}\right) \\ &\ge \frac{1}{p}\log\left(a^p\right) + \frac{1}{p'}\log\left(b^{p'}\right) \\ &= \log(a) + \log(b) = \log(ab).\end{aligned}$$

Since the logarithmic function is increasing, we are done (or we take exp on both sides).

□

Proof of theorem If $p = \infty$

$$\int_\Omega |fg| \le \|f\|_\infty \int_\Omega |g| = \|f\|_\infty \|g\|_1 .$$

The proof is similar for $p = 1$. So, assume now $1 < p < \infty$, $\|f\|_p = \|g\|_{p'} = 1$.

Integrating pointwise and applying Young's inequality almost everywhere, gives

$$\int_\Omega |f(x)g(x)|\,dx \le \int_\Omega \left(\frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'} \right) dx$$
$$= \frac{1}{p}\int_\Omega |f(x)|^p\,dx + \frac{1}{p'}\int_\Omega |g(x)|^{p'}\,dx$$
$$= \frac{1}{p} + \frac{1}{p'} = 1.$$

Now assuming $f, g \ne 0$ (else, we're done)

$$\int_\Omega \frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_{p'}}\,dx \le 1 \Rightarrow \int_\Omega |fg| \le \|f\|_p \|g\|_{p'}$$

Schwartz inequality $p = 2$

$$\left|\langle f, g\rangle_2\right| = \left|\int_\Omega f\overline{g}\right| \le \int_\Omega |fg| = \|fg\|_1 \le \|f\|_2 \|g\|_2 .$$

The L_p spaces not comparable on unbounded domains

Example We'll use $\Omega = \mathbb{R}$. Assume $0 < q < p < \infty$

Choose

$$f(x) := \begin{cases} 0 & |x| \le 1 \\ \dfrac{1}{|x|^{1/q}} & |x| > 1 \end{cases}$$

We have $f \in L_p(\mathbb{R})$, $f \notin L_q(\mathbb{R})$

Now choose

$$f(x) := \begin{cases} \dfrac{1}{|x|^{1/p}} & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

We have $f \in L_q(\mathbb{R})$, $f \notin L_p(\mathbb{R})$

Theorem If $|\Omega| < \infty$, $0 < q < p$, $f \in L_p(\Omega)$ then

$$\|f\|_{L_q(\Omega)} \le |\Omega|^{1/q-1/p} \|f\|_{L_p(\Omega)}.$$

Proof Define $r := p/q \ge 1$

$$\|f\|_q^q = \int_\Omega |f|^q = \int_\Omega |f|^q 1 \underbrace{\le}_{Holder} \left(\int_\Omega \left(|f|^q\right)^r\right)^{1/r} \left(\int_\Omega 1^{r'}\right)^{1/r'}$$

$$= \left(\int_\Omega |f|^p\right)^{q/p} |\Omega|^{1-q/p}$$

Theorem Minkowski for Lp spaces $1 \le p \le \infty$, $f, g \in L_p$,

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \ .$$

Proof for $1 < p < \infty$ ($p = 1, \infty$ is easier). W.l.g $f, g \ge 0$. We apply Hölder twice,

$$\begin{aligned}
\int (f+g)^p &= \int f\,(f+g)^{p-1} + \int g\,(f+g)^{p-1} \\
&\le \left(\|f\|_p + \|g\|_p\right)\left(\int (f+g)^{(p-1)p'}\right)^{1/p'} \\
&= \left(\|f\|_p + \|g\|_p\right)\left(\int (f+g)^{p}\right)^{1-1/p} \\
&= \left(\|f\|_p + \|g\|_p\right)\left(\int (f+g)^{p}\right)\underbrace{\left(\int (f+g)^{p}\right)^{-1/p}}_{\|f+g\|_p^{-1}} .
\end{aligned}$$

Theorem For $0 < p < 1$, we have

(i) $$\left\| \sum_k f_k \right\|_p^p \le \sum_k \|f_k\|_p^p$$

(ii) $\|f+g\|_p \le 2^{1/p-1}\left(\|f\|_p + \|g\|_p\right)$ or in general $\left\| \sum_{k=1}^N f_k \right\|_p \le N^{1/p-1} \sum_{j=1}^N \|f_k\|_p$

Proof The quasi-triangle inequality (ii) is derived from (i), by using $1 \le p^{-1} < \infty$,

$$\left\| \sum_{k=1}^N f_k \right\|_p \underset{(i)}{\le} \left(\sum_{k=1}^N \|f_k\|_p^p \right)^{1/p} = \left(\sum_{j=1}^N 1 \cdot \|f_k\|_p^p \right)^{1/p} \underset{\substack{\text{Discrete} \\ \text{Holder}}}{\le} \underbrace{\left(\sum_{k=1}^N 1^{\frac{1}{1-p}} \right)}_{N}^{(1-p)1/p} \left(\sum_{k=1}^N \|f_k\|_p \right) = N^{1/p-1} \sum_{k=1}^N \|f_k\|_p$$

To prove (i), we need the following lemma

Lemma I For $0 < p \le 1$ and any sequence of non-negative $a = \{a_k\}$,

$$\left(\sum_k a_k\right)^p \le \sum_k a_k^p$$

Proof We first prove $(a_1 + a_2)^p \le a_1^p + a_2^p$ and then apply induction.

To prove the inequality use $h(t) := t^p + 1 - (t+1)^p$. $h(0) = 0$ and $h'(t) = pt^{p-1} - p(t+1)^{p-1} \ge 0$. Therefore, $h(t) \ge 0$, for $t \ge 0$. This gives $t^p + 1 \ge (t+1)^p$. Setting $t = a_1 / a_2$ gives

$$\left(\frac{a_1}{a_2}\right)^p + 1 \ge \left(\frac{a_1}{a_2} + 1\right)^p \Rightarrow a_1^p + a_2^p \ge (a_1 + a_2)^p.$$

□

Proof of Theorem (i): Simply apply the lemma pointwise for $x \in \Omega$

$$\left\|\sum_k f_k\right\|_p^p \le \int_\Omega \left(\sum_k |f_k(x)|\right)^p dx \le \int_\Omega \left(\sum_k |f_k(x)|^p\right) dx = \sum_k \int_\Omega |f_k(x)|^p \, dx = \sum_k \|f_k\|_p^p.$$

□

Definition The space $l_p(\mathbb{Z})$, $0 < p \le \infty$, is the space of sequences $a = \{a_k\}_{k\in\mathbb{Z}}$, for which the norm is finite

$$\|a\|_{l_p} := \begin{cases} \left(\sum_k |a_k|^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_k |a_k|, & p = \infty. \end{cases}$$

Lemma II $l_p \subset l_q$ for $p \le q$. That is, for any sequence $a = \{a_k\}$

$$\|a\|_{l_q} \le \|a\|_{l_p} \quad .$$

Proof Case of $q = \infty$, for any $j \in \mathbb{Z}$,

$$|a_j| = \left(|a_j|^p\right)^{1/p} \le \left(\sum_k |a_k|^p\right)^{1/p} = \|a\|_{l_p} .$$

Therefore,

$$\|a\|_{l_\infty} = \sup_j |a_j| \le \|a\|_{l_p} .$$

For $q < \infty$, we have

$$\left(\sum_k |a_k|^q\right)^{p/q} \le \sum_k \left(|a_k|^q\right)^{p/q} = \sum_k |a_k|^p \Rightarrow \left(\sum_k |a_k|^q\right)^{1/q} \le \left(\sum_k |a_k|^p\right)^{1/p} .$$

$\square$

Hilbert spaces and $L_2(\Omega)$

Def Hilbert space H : Complete metric vector space induced by an inner product $\langle , \rangle : H \times H \to \mathbb{C}$.

Properties of the inner product:

i. symmetric $\langle f, g \rangle = \overline{\langle g, f \rangle}$,

ii. linear $\langle \alpha f_1 + \beta f_2, g \rangle = \alpha \langle f_1, g \rangle + \beta \langle f_2, g \rangle$,

iii. Positive definite $\langle f, f \rangle \geq 0$, with $\langle f, f \rangle = 0 \Leftrightarrow f = 0$.

The natural norm $\|f\|_H := \langle f, f \rangle^{1/2}$ satisfies

(i) Cauchy-Schwartz

$$|\langle f, g \rangle| \leq \|f\|_H \|g\|_H$$

(ii) Triangle inequality

$$\|f + g\|^2 = \|f\|^2 + 2\operatorname{Re}\langle f, g \rangle + \|g\|^2 \leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = \left(\|f\| + \|g\|\right)^2$$

So, a Hilbert space is a Banach space.

Examples

(i) $l_2(\mathbb{Z})$: $\langle \alpha, \beta \rangle_{l_2} := \sum_{i \in \mathbb{Z}} \alpha_i \overline{\beta}_i$, $\|\alpha\|_2 := \left(\sum_{i \in \mathbb{Z}} |\alpha_i|^2 \right)^{1/2}$

(ii) $L^2(\Omega)$: f, g measurable, $\langle f, g \rangle := C_\Omega \int_\Omega f(x) \overline{g(x)} dx$,

$$\|f\|_{L_2(\Omega)} = \|f\|_2 = \langle f, f \rangle^{1/2} = \left(C_\Omega \int_\Omega |f(x)|^2 \, dx \right)^{1/2} .$$

For $\Omega = \mathbb{R}^n, C_\Omega = 1$. For $\Omega = [-\pi, \pi]^n, C_\Omega = \frac{1}{(2\pi)^n}$.

Multivariate algebraic polynomials

We define $\Pi_{r-1}\left(\mathbb{R}^n\right)$: polynomials of degree $r-1$.

Let $\alpha = \left(\alpha_1, \ldots, \alpha_n\right) \in \mathbb{Z}_+^n$, $|\alpha| := \sum_{i=1}^{n} \alpha_i$.

Monomial $x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i}$, $x = \left(x_1, \ldots, x_n\right) \in \mathbb{R}^n$.

Polynomial $P \in \Pi_{r-1}$

$$P\left(x\right) = \sum_{|\alpha|<r} a_\alpha x^\alpha$$

Spaces of smooth functions

Multivariate derivatives: A partial derivative of order r

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, \quad D^\alpha f = \frac{\partial^r f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| := \sum_{i=1}^n \alpha_i = r.$$

Definition $C^r(\Omega)$: The space of all continuously differentiable functions of order r in the classical sense.

$$\|f\|_{C^r(\Omega)} := \sum_{|\alpha| \le r} \|D^\alpha f\|_{L_\infty(\Omega)},$$

The ***semi-norm***

$$|f|_{C^r(\Omega)} := \sum_{|\alpha| = r} \|D^\alpha f\|_\infty$$

Examples $C^r(\mathbb{R})$ Then $\|f\|_{C^r(\mathbb{R})} = \sum_{k=0}^r \left\|f^{(k)}\right\|_\infty$ is a norm $|f|_{C^r(\mathbb{R})} = \left\|f^{(r)}\right\|_\infty$ is a semi-norm with the polynomials of degree $r-1$ as a null-space

Sobolev spaces

Definition We define the space of ***test-functions*** $C_0^r(\Omega)$ - continuously r-differentiable with compact support in Ω.

Definition Sobolev spaces $W_p^r(\Omega)$, $1 \le p \le \infty$

Def I For $1 \le p < \infty$, completion of $C_0^r(\Omega)$ with respect to the norm $\sum_{|\alpha| \le r} \left\| \partial^\alpha f \right\|_{L_p(\Omega)}$. For $p = \infty$, we take $W_\infty^r(\Omega) := C^r(\Omega)$.

Def II Let $f \in L_p(\Omega)$. Now for $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le r$, $g := \partial^\alpha f$ is the ***distributional (generalized) derivative*** of f if it is a function and for all $\phi \in C_0^r(\Omega)$

$$\int_\Omega g\phi = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \phi .$$

The Sobolev norm and semi-norm. We require that the distributional derivatives exist as functions(!) in $L_p(\Omega)$ and

$$\|f\|_{W_p^r(\Omega)} := \sum_{|\alpha| \le r} \left\| \partial^\alpha f \right\|_{L_p(\Omega)} < \infty \qquad |f|_{W_p^r(\Omega)} := \sum_{|\alpha| = r} \left\| \partial^\alpha f \right\|_{L_p(\Omega)} .$$

Theorem $W_p^r(\Omega)$ is a Banach space

Modulus of smoothness

Def The ***difference operator*** Δ_h^r. For $h \in \mathbb{R}^n$ we define $\Delta_h(f,x) = f(x+h) - f(x)$. For general $r \ge 1$ we define

$$\Delta_h^r(f,x) = \underbrace{\Delta_h \circ \cdots \Delta_h}_{r}(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh).$$

Remarks

1. For $\Omega \subset \mathbb{R}^n$, we modify to $\Delta_h^r(f,x) := \Delta_h^r(f,x,\Omega)$, where $\Delta_h^r(f,x) = 0$, in the case $[x, x+rh] \not\subset \Omega$. So for $\Omega = [a,b]$, $\Delta_h^r(f,x) = 0$ on $[b-rh, b]$, for any function.
2. As an operator on $L_p(\Omega)$, $1 \le p \le \infty$, we have that $\left\|\Delta_h^r\right\|_{L_p \to L_p} \le 2^r$. Assume $\Omega = \mathbb{R}^n$, then

$$\left\|\Delta_h^r(f,\bullet)\right\|_p \le \sum_{k=0}^{r} \binom{r}{k} \left\|f(\bullet + kh)\right\|_p = \sum_{k=0}^{r} \binom{r}{k} \|f\|_p = 2^r \|f\|_p$$

Def The ***modulus of smoothness*** of order r of a function $f \in L_p(\Omega)$, $0 < p \le \infty$, at the parameter $t > 0$

$$\omega_r(f,t)_p := \sup_{|h| \le t} \left\|\Delta_h^r(f,x)\right\|_{L_p(\Omega)}.$$

Example non continuous function. Let $\Omega = [-1,1]$. $f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \end{cases}$

Let's compute $\omega_r(f,t)_{L_p([-1,1])}$, $0 < t < 1$. For $0 < h \le t$

$$\Delta_h(f,x) = \begin{cases} 0 & -1 \le x \le -h \\ 1 & -h < x \le 0 \\ 0 & 0 < x \le 1 \end{cases}$$

For $p = \infty$ we get $\omega_1(f,t)_{L_\infty([-1,1])} = \sup_{|h| \le t} \|\Delta_h f\|_{L_\infty([-1,1])} = 1$.

For $p \ne \infty$ we get $\omega_1(f,t)_{L_p([-1,1])} = \sup_{|h| \le t} \|\Delta_h f\|_{L_p([-1,1])} = t^{1/p}$.

$$\Delta_h^2(f,x) = \Delta_h(\Delta_h f, x) = \begin{cases} 0 & -1 \le x \le -2h \\ 1 & -2h < x \le -h \\ -1 & -h < x \le 0 \\ 0 & 0 \le x \le 1 \end{cases}$$

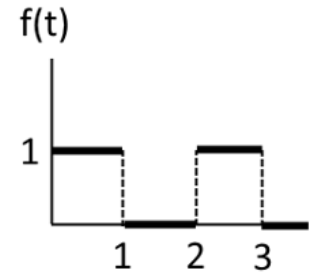
We get $\omega_2(f,t)_{L_p([-1,1])} = (2t)^{1/p}$

In general, we get $\omega_r(f,t)_{L_p([-1,1])} \le C(r,p)t^{1/p}$

Quick jump ahead (Generalized Lipschitz / Besov smoothness) … for $\alpha \le 1/\tau$, $r = \lfloor \alpha \rfloor + 1$,

$$|f|_{B^{\alpha}_{\tau,\infty}} := \sup_{t>0} t^{-\alpha}\omega_r(f,t)_\tau \le \sup_{0<t\le 2} t^{-\alpha}\omega_r(f,t)_\tau \le c \sup_{0<t\le 2} t^{1/\tau-\alpha} < \infty .$$

We then say that f has α (weak-type) smoothness. Observe that in this example α can be arbitrarily large as long as the integration takes place with τ sufficiently small.



Machine learning perspective Let f be a ‘binary classification’ step function with M steps.

You will compute (assignment I) for $0 < \alpha < 1$, $|f|_{B^{\alpha}_{\tau,\infty}} \sim (2M)^{1/\tau}$.

- The feature space is `problematic’ for a simple ML model such as logistic regression.
- As a discontinuous function, ‘simpler’ smoothness function spaces do not contain it.
- Decision trees will find the clusters, so no need for DL.
- DL? For $M = 2^j$, the function can be realized/learnt by a neural network with $\sim j$ blocks,
- Each block has 4 neurons/features (2 layers with 2 neurons each)
- After the k-th block the function f_k has 2^{j-k} ‘steps’ with $|f_k|_{B^{\alpha}_{\tau,\infty}} \sim 2^{(j-k)/\tau}$.
- We can realize that the last representation layer as $\tilde{f}(\tilde{x}) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$, so it can be easily consumed by a logistic model.

Properties

1. $\omega_r(f,t)_p \le 2^r \|f\|_{L_p(\Omega)}$, $1 \le p \le \infty$.

2. $\omega_r(f,t)_p$ is non-decreasing in t

3. For $1 \le p \le \infty$ the ***sub-linearity*** property

$$\begin{aligned}
\left|\Delta_h^r(f+g,x)\right| &= \left|\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}(f+g)(x+kh)\right| \\
&\le \left|\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k} f(x+kh)\right| + \left|\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k} g(x+kh)\right| \\
&= \left|\Delta_h^r(f,x)\right| + \left|\Delta_h^r(g,x)\right|.
\end{aligned}$$

gives

$$\omega_r(f+g,t)_p \le \omega_r(f,t)_p + \omega_r(g,t)_p .$$

4. For $N \geq 1$, $\omega_r(f, Nt)_p \leq N^r \omega_r(f,t)_p$, $1 \leq p \leq \infty$. We prove this using the property (**assignment**)

$$\Delta_{Nh}^r(f,x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta_h^r(f, x + k_1 h + \cdots + k_r h).$$

Let's see the case $r=1$,

$$\begin{aligned}
\Delta_{Nh}(f,x) &= f(x+Nh) - f(x) \\
&= f(x+Nh) - f\left(x+(N-1)h\right) + f\left(x+(N-1)h\right) - \cdots + f(x+h) - f(x) \\
&= \sum_{k=0}^{N-1} \Delta_h(f, x+kh)
\end{aligned}$$

Then, for any $h \in \mathbb{R}^n$, $|h| \leq t$

$$\begin{aligned}
\left\|\Delta_{Nh}^r(f,\cdot)\right\|_p &\leq \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \left\|\Delta_h^r(f, \cdot + k_1 h + \cdots + k_r h)\right\|_p \\
&= \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \left\|\Delta_h^r(f,\cdot)\right\|_p \leq N^r \omega_r(f,t)_p.
\end{aligned}$$

Taking supremum over all $h \in \mathbb{R}^n$, $|h| \leq t$, gives $\omega_r(f, Nt)_p \leq N^r \omega_r(f,t)_p$. It is easy to see that for $0 < p < 1$, the same proof yields $\omega_r(f, Nt)_p \leq N^{r/p} \omega_r(f,t)_p$.

5. From (4) we get for $1 \le p \le \infty$,

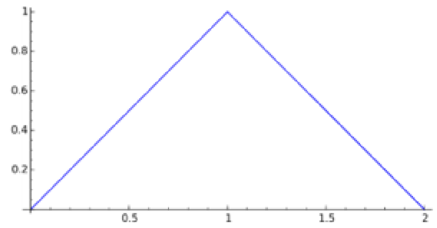
$$\omega_r(f,\lambda t)_p \le (\lambda+1)^r \omega_r(f,t)_p, \qquad \lambda > 0$$

proof $\omega_r(f,\lambda t)_p \le \omega_r(f,\lfloor \lambda+1 \rfloor t)_p \le (\lfloor \lambda+1 \rfloor)^r \omega_r(f,t)_p \le (\lambda+1)^r \omega_r(f,t)_p$.

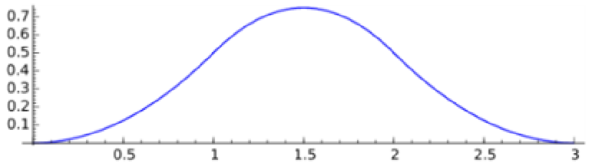
Theorem [connection between Sobolev and modulus] For $g \in W_p^r(\Omega)$, $1 \le p \le \infty$, we have that

$$\omega_r(g,t)_{L_p(\Omega)} \le C(r,n) t^r |g|_{W_p^r(\Omega)}, \qquad \forall t > 0.$$

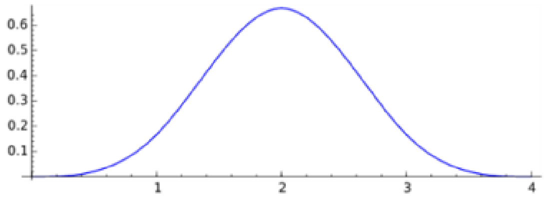
Proof for $\Omega = \mathbb{R}$. Recall the B-Splines, $N_1 = \mathbf{1}_{[0,1]^n}$. In general, $N_r := N_{r-1} * N_1 = \int_{\mathbb{R}^n} N_{r-1}(x-t) N_1(t)\,dt$.



N_2



N_3



N_4

- Properties:
  - Order r
  - Support $[0,r]^n$
  - Piecewise polynomial of degree $r-1$ with breakpoints (knots) at the integers
  - Smoothness $r-2$, thus in Sobolev W_p^{r-1}.
  - Tensor-product in multivariate case $N_r(x) := \tilde{N}_r(x_1) \times \cdots \times \tilde{N}_r(x_n)$, where $\tilde{N}_r$ is the univariate B-spline.
  - $\int_{\mathbb{R}^n} N_r(x)\,dx = 1$

Here, we use the fact that for $h \in \mathbb{R}^n$, $\left|\Delta_{-h}^r(f,x)\right| = \left|\Delta_h^r(f,x-rh)\right|$. So, w.l.g., for any $t>0$, we can work with $0<h\le t$. Define $N_r(x,h) := h^{-1}N_r\left(h^{-1}x\right)$, $h>0$. Let $g \in C^1(\mathbb{R})$. Then

$$\begin{aligned}
h^{-1}\Delta_h(g,x) &= h^{-1}\left(g(x+h)-g(x)\right)\\
&= h^{-1}\int_x^{x+h} g'(u)\,du\\
&= \int_{\mathbb{R}} g'(x+u)\,N_1(u,h)\,du
\end{aligned}$$

We claim that for $g \in C^r(\mathbb{R})$

$$h^{-r}\Delta_h^r(g,x) = \int_{\mathbb{R}} g^{(r)}(x+u)N_r(u,h)\,du$$

To see this, we apply induction

$$\begin{aligned}
h^{-r}\Delta_h^r(g,x) &= h^{-1}h^{-(r-1)}\left(\Delta_h^{r-1}(g,x+h)-\Delta_h^{r-1}(g,x)\right)\\
&= h^{-1}\left(\int_{\mathbb{R}} g^{(r-1)}(x+h+u)N_{r-1}(u,h)\,du - \int_{\mathbb{R}} g^{(r-1)}(x+u)N_{r-1}(u,h)\,du\right)\\
&= h^{-1}\int_x^{x+h}\int_{-\infty}^{\infty} g^{(r)}(v+u)N_{r-1}(u,h)\,du\,dv\\
&= \int_{-\infty}^{\infty} N_{r-1}(u,h)\left(h^{-1}\int_x^{x+h} g^{(r)}(v+u)\,dv\right)du
\end{aligned}$$

$$= \int_{-\infty}^{\infty} N_{r-1}(u,h) \left(h^{-1} \int_{x}^{x+h} g^{(r)}(v+u)dv \right) du$$

$$= \int_{-\infty}^{\infty} N_{r-1}(u,h) \left(\int_{-\infty}^{\infty} g^{(r)}(v+u) N_1(v-x,h) dv \right) du$$

$$\overset{v+u=x+y}{=} \int_{-\infty}^{\infty} N_{r-1}(u,h) \left(\int_{-\infty}^{\infty} g^{(r)}(x+y) N_1(y-u,h) dy \right) du$$

$$= \int_{-\infty}^{\infty} g^{(r)}(x+y) \left(\int_{-\infty}^{\infty} N_{r-1}(u,h) N_1(y-u,h) du \right) dy$$

$$= \int_{-\infty}^{\infty} g^{(r)}(x+y) \mathrm{N}_r(y,h) \mathrm{dy}$$

Now, let us see the proof for $p=1$. Assume $g \in W_1^r(\mathbb{R}) \cap C^r(\mathbb{R})$. Let $0<h \le t$

$$\begin{aligned}\int_{\mathbb{R}} \left|\Delta_h^r(g,x)\right| dx &\le h^r \int_{\mathbb{R}}\int_{\mathbb{R}} \left|g^{(r)}(x+u)\right| \left|N_r(u,h)\right| du dx \\ &\le h^r \int_{\mathbb{R}} \left|N_r(u,h)\right| du \int_{\mathbb{R}} \left|g^{(r)}(x+u)\right| dx \\ &\le t^r \int_{\mathbb{R}} \left|g^{(r)}(x)\right| dx \\ &\le t^r \left|g\right|_{W_1^r(\mathbb{R})}.\end{aligned}$$

For general $1 \le p < \infty$ we need Minkowski's inequality. It says that for measurable non-negative functions $\varphi : B \to \mathbb{R},\ \rho : A \times B \to \mathbb{R}$

$$\left\{\int_A \left(\int_B \varphi(y)\rho(x,y)dy\right)^p dx\right\}^{1/p} \le \int_B \varphi(y)\left(\int_A \rho(x,y)^p dx\right)^{1/p} dy$$

Or written differently (as an integral generalization of the 'discrete' Minkowski inequality)

$$\left\|\int_B \varphi(y)\rho(\cdot,y)dy\right\|_{L_p(A)} \le \int_B \varphi(y)\left\|\rho(\cdot,y)\right\|_{L_p(A)} dy \qquad \Leftrightarrow \left\|\sum_k \rho_k(\cdot)\right\|_p \le \sum_k \left\|\rho_k(\cdot)\right\|_p$$

Using it we have for $g \in W_p^r(\mathbb{R}) \bigcap C^r(\mathbb{R})$

$$\begin{aligned}\int_{\mathbb{R}} \left|\Delta_h^r(g,x)\right|^p dx &\le h^{pr} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|g^{(r)}(x+u)\right|\left|N_r(u,h)\right| du\right)^p dx \\ &\le h^{pr} \left(\int_{\mathbb{R}} \left|N_r(u,h)\right| \left\|g^{(r)}(\cdot+u)\right\|_{L_p(\mathbb{R})} du\right)^p \\ &\le h^{pr} \left(\int_{\mathbb{R}} \left|N_r(u,h)\right| \left\|g^{(r)}\right\|_{L_p(\mathbb{R})} du\right)^p \\ &\le t^{pr} \left\|g^{(r)}\right\|_{L_p(\mathbb{R})}^p \\ &= t^{pr} \left|g\right|_{W_p^r(\mathbb{R})}^p .\end{aligned}$$

For a general function $g \in W_p^r(\mathbb{R})$ we use the density of $C^r(\mathbb{R}) \bigcap W_p^r(\mathbb{R})$ in $W_p^r(\mathbb{R})$

Corollary For any $P \in \Pi_{r-1}(\mathbb{R})$, $P(x) = \sum_{k=0}^{r-1} a_k x^k$,

$$h^{-r}\Delta_h^r(P,x) = \int_{\mathbb{R}} P^{(r)}(x+u) N_r(u,h)\, du = 0 \Rightarrow \Delta_h^r(P,x) = 0 \Rightarrow \omega_r(P,t)_p = 0$$

Marchaud inequalities

We know that for any $1 \le k < r$, $1 \le p \le \infty$,

$$\omega_r(f,t)_p = \sup_{|h|\le t} \left\| \Delta_h^r(f) \right\|_p = \sup_{|h|\le t} \left\| \Delta_h^{r-k} \Delta_h^k(f) \right\|_p \le 2^{r-k} \sup_{|h|\le t} \left\| \Delta_h^k(f) \right\|_p = 2^{r-k} \omega_k(f,t)_p .$$

The direct inverse cannot be true. If we take $\Omega = [a,b]$ and a polynomial $P \in \Pi_{r-1}$, then $\omega_r(P,t)_p = 0$, but we don't necessarily have $\omega_k(P,t)_p = 0$ for $0 \le k < r$.

Theorem. For any $1 \le k < r$, $1 \le p \le \infty$,

On $\Omega = \mathbb{R}$,
$$\omega_k(f,t)_p \le ct^k \int_t^\infty \frac{\omega_r(f,s)_p}{s^{k+1}} ds, \qquad t > 0.$$

On $\Omega = [a,b]$,
$$\omega_k(f,t)_p \le ct^k \left(\int_t^{b-a} \frac{\omega_r(f,s)_p}{s^{k+1}} ds + \frac{\|f\|_p}{(b-a)^k} \right), \quad 0 < t \le \frac{b-a}{r}.$$

Lip spaces

Def For a domain $\Omega \subset \mathbb{R}^n$ and $0 < \alpha \le 1$, we shall say that $f \in Lip(\alpha) = Lip(\alpha, \infty)$, if there exists $M > 0$, such that $|f(x) - f(y)| \le M|x-y|^\alpha$, for all $x, y \in \Omega$. We shall denote $|f|_{Lip(\alpha)}$ by the infimum over all M satisfying the condition. Observe that we can replace the condition by

$$|\Delta_h(f,x)| \le M|h|^\alpha,\ \forall h \in \mathbb{R}^n \Rightarrow$$
$$\omega_1(f,t)_\infty \le Mt^\alpha,\ \forall t > 0 \Rightarrow$$
$$t^{-\alpha}\omega_1(f,t)_\infty \le M,\ \forall t > 0.$$

For $1 \le p \le \infty$, we define

$$|f|_{lip(\alpha,p)} := \sup_{t>0} t^{-\alpha}\omega_1(f,t)_p \ .$$

Example For $f(x)=x^{\alpha}$, $0<\alpha\le 1$, $f\in Lip(\alpha)$, $f\notin Lip(\beta)$, $\beta>\alpha$.

Proof

(i) Assume $f\in Lip(\beta)$, $\beta>\alpha$. Then for $0<x\le 1$,

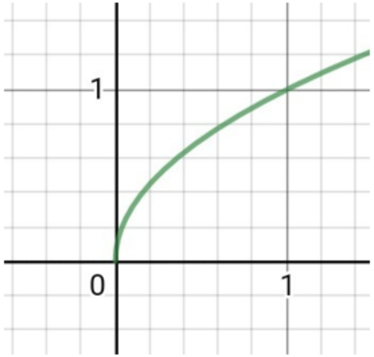
$$x^{\alpha}-0^{\alpha}=x^{\alpha}\le M(x-0)^{\beta}=Mx^{\beta}\Rightarrow x^{\alpha-\beta}\le M\Rightarrow \text{contradiction}$$

(ii) We use the inequality $(a+b)^{\alpha}\le a^{\alpha}+b^{\alpha}$. Assume w.l.g $x\ge y$, we set $a=y, b=x-y$ and obtain

$$x^{\alpha}\le y^{\alpha}+(x-y)^{\alpha}\Rightarrow x^{\alpha}-y^{\alpha}\le (x-y)^{\alpha},\ |f|_{Lip(\alpha)}=1.$$

□

However, for any $0<\alpha\le 1$, $f(x)=x^{\alpha}\in Lip(1,1)$, because



$$\int_0^1 |f'(x)|dx=1\Rightarrow f'\in L_1\Rightarrow f\in W_1^1([0,1])$$

$$\Rightarrow \omega_1(f,t)_1\le t|f|_{1,1}=t,\quad \forall t>0$$

$$\Rightarrow |f|_{Lip(1,1)}=\sup_{t>0} t^{-1}\omega_1(f,t)_1=1.$$

Generalized Lip are a special case of Besov spaces. For any $\alpha>0$, let $r:=\lfloor \alpha \rfloor+1$,

$$|f|_{B^{\alpha}_{p,\infty}}:=\sup_{t>0} t^{-\alpha}\omega_r(f,t)_p .$$

Approximation using uniform piecewise constants (numerical integration)

The B-Spline of order one (degree zero, smoothness -1) $N_1(x) = \mathbf{1}_{[0,1]}(x)$.

Let $\Omega = \mathbb{R}$ or $\Omega = [a,b]$. We approximate from the space

$$S(N_1)^h := \left\{ \sum_{k\in\mathbb{Z}} c_k N_1\left(h^{-1}x - k\right) \right\} = \left\{ \sum_{k\in\mathbb{Z}} c_k \mathbf{1}_{[kh,(k+1)h]}(x) \right\}.$$

Theorem For $f \in W_p^1(\mathbb{R})$, $1 \le p \le \infty$,

$$E\left(f, S(N_1)^h\right)_{L_p(\mathbb{R})} := \inf_{g\in S(N_1)^h} \|f-g\|_{L_p(\mathbb{R})} \le h|f|_{W_p^1(\mathbb{R})}.$$

Proof First assume $f \in C^1(\mathbb{R}) \cap W_p^1(\mathbb{R})$. Let's take the interval $[kh,(k+1)h]$. Then, for $p = \infty$

$$|f(x) - f(kh)| = \left| \int_{kh}^{x} f'(u)\,du \right| \le h \max_{kh\le u\le(k+1)h} |f'(u)|.$$

So, select $c_k := f(kh)$ and you get the theorem for $p = \infty$. For $1 \le p < \infty$ we do something similar

$$|f(x) - f(kh)|^p \le \left(\int_{kh}^{(k+1)h} |f'(u)|\,du \right)^p, \qquad x \in [kh,(k+1)h].$$

Then

$$\int_{kh}^{(k+1)h} |f(x) - f(kh)|^p \, dx \le h \left(\int_{kh}^{(k+1)h} |f'(u)| \, du \right)^p$$

$$\underset{Holder}{\le} h \left(\|f'\|_{L_p([kh,(k+1)h])} \|1\|_{L_{p'}([kh,(k+1)h])} \right)^p$$

$$= h h^{p/p'} \|f'\|_{L_p([kh,(k+1)h])}^p$$

$$= h^p \|f'\|_{L_p([kh,(k+1)h])}^p .$$

$$1 + \frac{p}{p'} = 1 + p\left(1 - \frac{1}{p}\right)$$

$$= 1 + p - 1 = p$$

Therefore, with $g(x) := \sum_k f(kh) N_1(h^{-1}x - k)$, we get

$$\|f - g\|_p^p = \int_{-\infty}^{\infty} |f(x) - g(x)|^p \, dx = \sum_k \int_{kh}^{(k+1)h} |f(x) - f(kh)|^p \, dx \le \sum_k h^p \|f'\|_{L_p([kh,(k+1)h])}^p = h^p \|f'\|_p^p .$$

Now assume $f \in W_p^1(\mathbb{R})$, $1 \le p < \infty$. There exist sequences $\{f_k\}$, $f_k \in C^1(\mathbb{R}) \cap W_p^1(\mathbb{R})$, $\{g_k\}$, $g_k \in S(N_1)^h$, such that $\|f - f_k\|_{W_p^1(\mathbb{R})} \underset{k\to\infty}{\to} 0$ and $\|f_k - g_k\|_{L_p(\mathbb{R})} \le h|f_k|_{W_p^1(\mathbb{R})}$. This gives

$$\begin{aligned}\|f - g_k\|_p &\le \|f - f_k\|_p + \|f_k - g_k\|_p \\ &\le \|f - f_k\|_p + h|f_k|_{1,p} \underset{k\to\infty}{\to} 0 + h|f|_{1,p}\end{aligned}$$

□

Linear approximation of Lip functions

Why linear? There is a 'near best' (possibly up to a constant) linear realization of the approximation.

Theorem: Let $f \in Lip(\alpha)$. Approximation with uniform piecewise constants gives

$$E_N(f)_{L_\infty([0,1])} := \inf_{\phi \in S(N_1)^{1/N}} \|f - \phi\|_\infty \leq C N^{-\alpha} |f|_{Lip(\alpha)} .$$

Inverse Theorem: Assume $E_N(f)_\infty \leq M N^{-\alpha}$, $N \geq 1$. Then, $f \in Lip(\alpha)$.

Example $E_N(x^\alpha) \sim N^{-\alpha}$, $0 < \alpha \leq 1$.

Non-linear approximation of Lip functions

$$\Sigma_N := \left\{ \sum_{j=0}^{N-1} c_j \mathbf{1}_{[t_j, t_{j+1})} : T = \{t_j\},\ 0 = t_0 < t_1 < \cdots < t_N = 1 \right\} , \quad \sigma_N(f)_p := \inf_{g \in \Sigma_N} \|f - g\|_p .$$

<u>This is the theoretical model of a univariate decision tree!</u>

$$Var(f) := \sup_T \left\{ \sum \left| f(t_{j+1}) - f(t_j) \right| \right\} .$$

If f' exists a.e., $Var(f) = \|f'\|_1$. Why?

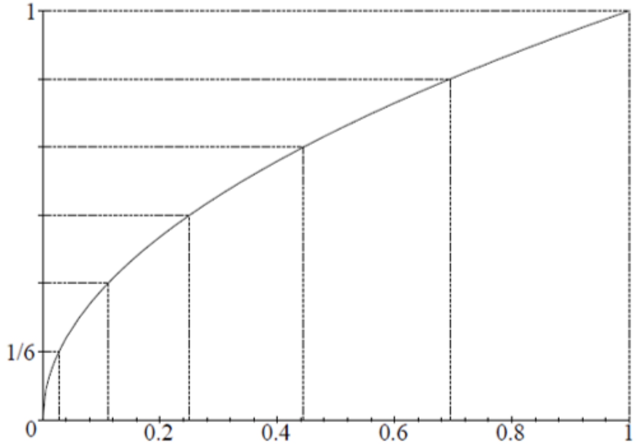
$$\int_0^1 |f'(x)| dx = \lim_{h \to 0} \sum_k h \frac{\left| f((k+1)h) - f(kh) \right|}{h} .$$

Let's go back to the examples $f(x) = x^{\alpha}$. In our case $\|f'\|_1 = \int_0^1 f'(x) dx = f(1) - f(0) = 1$.

Now, create a partition where $Var_{[t_j, t_{j+1}]}(f) \le \dfrac{Var(f)}{N}$. If a_j is the median value in $[t_j, t_{j+1}]$, then

$$\left| f(x) - a_j \right| \le \frac{Var_{[t_j, t_{j+1}]}(f)}{2} \le \frac{Var(f)}{2N}, \ \forall x \in [t_j, t_{j+1}] .$$

For $f(x)=x^{\alpha}$, $0<\alpha\le 1$, this gives a free knot spline $g\in\Sigma_N$ with



$$\|f-g\|_{\infty}\le\frac{Var(f)}{2N}\le\frac{1}{2N} \ .$$

To obtain an equidistant partition of the range, we choose

$$t_j=\left(\frac{j}{N}\right)^{1/\alpha} .$$

We already saw that $f\in W_1^1$. This implies $f\in Lip(1,1)$, because

$$\sup_{t>0} t^{-1}\omega_1(f,t)_1\le\sup_{t>0} ct^{-1}t|f|_{1,1}=c|f|_{1,1}<\infty$$

So, we see the advantage of nonlinear approximation for the family $f(x)=x^{\alpha}$, $0<\alpha<1$,

$$f\in Lip(\alpha,\infty)\Rightarrow E_N(f)_{\infty}\sim N^{-\alpha} \ , \quad f\in Lip(1,1)\Rightarrow\sigma_N(f)_{\infty}\sim N^{-1}.$$

Besov Spaces

Let $\alpha > 0$, $0 < q, p \le \infty$. Let $r \ge \lfloor \alpha \rfloor + 1$. The Besov space $B_q^\alpha\left(L_p(\Omega)\right)$ is the collection of functions $f \in L_p(\Omega)$ for which

$$|f|_{B_q^\alpha(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty \left[t^{-\alpha} \omega_r(f,t)_p \right]^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\alpha} \omega_r(f,t)_p, & q = \infty. \end{cases}$$

is finite. The norm is

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + |f|_{B_q^\alpha(L_p(\Omega))}.$$

Why are we asking for the condition $r \ge \lfloor \alpha \rfloor + 1$? Otherwise, the space is 'trivial'

Theorem (univariate case) For $r < \alpha$, $1 \le p \le \infty$, we get that $B_q^\alpha\left(L_p(\Omega)\right) = \Pi_{r-1}$ if $\Omega = [a,b]$ and $B_q^\alpha\left(L_p(\Omega)\right) = \{0\}$ if $\Omega = \mathbb{R}$.

Theorem The space $B_q^\alpha\left(L_p(\Omega)\right)$ does not depend on the choice of $r \ge \lfloor \alpha \rfloor + 1$ (application of the Marchaud inequality).

Theorem For a bounded domain we can equivalently integrate the semi-norm on $[0,1]$. That is,

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \begin{cases} \left(\int_0^1 \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}, & 0<q<\infty, \\ \sup\limits_{0<t\le 1} t^{-\alpha}\omega_r(f,t)_p, & q=\infty. \end{cases}$$

Proof If Ω is bounded, then we have $\omega_r(f,t)_p \equiv const$ for $t \ge diam(\Omega)$. Therefore for $1/2 \le t \le \infty$,

$$\omega_r(f,1/2)_p \le \omega_r(f,t)_p \le \omega_r(f,diam(\Omega))_p = \omega_r\left(f,\frac{2diam(\Omega)}{2}\right)_p \le (1+2diam(\Omega))^r \omega_r(f,1/2)_p.$$

This gives

$$\begin{aligned} \int_1^\infty \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t} &\le C\left(\omega_r(f,1/2)_p\right)^q \int_1^\infty t^{-q\alpha-1}dt \\ &\le C\left(\omega_r(f,1/2)_p\right)^q \\ &\le C(\alpha,q,\Omega)\int_{1/2}^1 \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t} \end{aligned}$$

Lemma For any domain taking the integral over $[0,1]$ gives a quasi-norm equivalent to $\|f\|_{B_q^\alpha(L_p(\Omega))}$

Proof We replace the integral over $[1,\infty]$ by

$$\int_1^\infty \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t} \le C\|f\|_p^q \int_1^\infty t^{-q\alpha-1}dt$$
$$= C(\alpha,q)\|f\|_p^q .$$

Therefore

$$\|f\|_{B_q^\alpha(L_p(\Omega))} \sim \|f\|_p + \left(\int_0^1 \left[t^{-\alpha}\omega_r(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q} .$$

Theorem $B_{q_1}^{\alpha_1}(L_p) \subseteq B_{q_2}^{\alpha_2}(L_p)$ if $\alpha_2 < \alpha_1$.

Proof $(q_1 = q_2)$ We may use $r_1 = \lfloor \alpha_1 \rfloor + 1 \geq \lfloor \alpha_2 \rfloor + 1 = r_2$ to equivalently define $B_{q_2}^{\alpha_2}(L_p)$

For $0 < t \leq 1$, $t^{-\alpha_2} \leq t^{-\alpha_1}$. So,

$$\|f\|_{B_q^{\alpha_2}(L_p)} \leq C\left(\|f\|_p + \left(\int_0^1 \left[t^{-\alpha_2}\omega_{r_1}(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}\right)$$

$$\leq C\left(\|f\|_p + \left(\int_0^1 \left[t^{-\alpha_1}\omega_{r_1}(f,t)_p\right]^q \frac{dt}{t}\right)^{1/q}\right)$$

$$\leq C\|f\|_{B_q^{\alpha_1}(L_p)}$$

Theorem $W_p^m \subseteq B_q^\alpha\left(L_p\right)$, $\forall \alpha < m$, $1 \le p \le \infty$, $0 < q \le \infty$.

Proof Let $g \in W_p^m(\Omega)$. This implies $g \in L_p(\Omega)$. We have that $r := \lfloor \alpha \rfloor + 1 \le m$. It is sufficient to take the integral over $[0,1]$.

$$\begin{aligned}
\int_0^1 \left[t^{-\alpha}\omega_r(g,t)_p\right]^q \frac{dt}{t} &\le C\int_0^1 \left[t^{-\alpha}t^r |g|_{r,p}\right]^q \frac{dt}{t} \\
&\le C|g|_{r,p}^q \int_0^1 t^{(r-\alpha)q-1}dt \\
&\le C|g|_{r,p}^q .
\end{aligned}$$

Discretization of the Besov semi-norm

Theorem One has the following equivalent form of the Besov semi-norm

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \begin{cases} \left(\sum_{k=-\infty}^{\infty} \left[2^{k\alpha} \omega_r\left(f, 2^{-k}\right)_p \right]^q \right)^{1/q}, & 0 < q < \infty. \\ \sup_{k \in \mathbb{Z}} 2^{k\alpha} \omega_r\left(f, 2^{-k}\right)_p, & q = \infty. \end{cases}$$

Proof Define $\varphi(t) := t^{-\alpha} \omega_r(f,t)_p$. Then we claim that for $t \in \left[2^{-k-1}, 2^{-k}\right]$, $k \in \mathbb{Z}$, we have

$$2^{-r} \varphi\left(2^{-k}\right) \le \varphi(t) \le 2^{\alpha} \varphi\left(2^{-k}\right).$$

To see that, we use the following properties:

(i) $\omega_r(f,t)_p$ is non-decreasing

(ii) For $N \in \mathbb{N}$, $1 \le p \le \infty$, $\omega_r(f, Nt)_p \le N^r \omega_r(f,t)_p$

The left-hand side

$$2^{-r}\varphi\left(2^{-k}\right)=2^{k\alpha-r}\omega_r\left(f,2^{-k}\right)_p=2^{k\alpha-r}\omega_r\left(f,22^{-k-1}\right)_p$$

$$\underbrace{\le}_{\text{(ii)}}2^{k\alpha-r}2^r\omega_r\left(f,2^{-k-1}\right)_p\underbrace{\le}_{\text{(i)}}2^{k\alpha}\omega_r\left(f,t\right)_p\le t^{-\alpha}\omega_r\left(f,t\right)_p$$

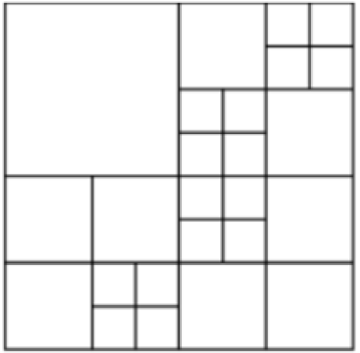
The right-hand side

$$t^{-\alpha}\omega_r\left(f,t\right)_p\underbrace{\le}_{\text{(i)}}t^{-\alpha}\omega_r\left(f,2^{-k}\right)_p\le 2^{(k+1)\alpha}\omega_r\left(f,2^{-k}\right)_p\le 2^{\alpha}\varphi\left(2^{-k}\right)$$

This gives us for $0<q<\infty$, $k\in\mathbb{Z}$

$$\int_{2^{-k-1}}^{2^{-k}}\varphi(t)^q\frac{dt}{t}\sim\varphi\left(2^{-k}\right)^q\int_{2^{-k-1}}^{2^{-k}}\frac{dt}{t}\sim\varphi\left(2^{-k}\right)^q\Rightarrow\int_{2^{-k-1}}^{2^{-k}}\left(t^{-\alpha}\omega_r\left(f,t\right)_p\right)^q\frac{dt}{t}\sim\left[2^{k\alpha}\omega_r\left(f,2^{-k}\right)_p\right]^q.$$

Discretization over cubes



Definition [Dyadic cubes] Let $D := \{D_k : k \in \mathbb{Z}\}$

$$D_k := \{Q = 2^{-k}[m_1, m_1 + 1] \times \cdots \times [m_n, m_n + 1] : m \in \mathbb{Z}^n\}.$$

Observe that $Q \in D_k \Rightarrow |Q| = 2^{-kn}$.

For nonlinear/adaptive/sparse approximation in $L_p(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, it is useful to use the special cases of Besov spaces

$$B_\tau^\alpha := B_\tau^\alpha(L_\tau(\Omega)), \qquad \frac{1}{\tau} = \frac{\alpha}{n} + \frac{1}{p}.$$

Theorem $\Omega = \mathbb{R}^n$. We have the equivalence

$$|f|_{B_\tau^\alpha} \sim \left(\sum_{k \in \mathbb{Z}} \left(2^{k\alpha} \omega_r(f, 2^{-k})_\tau \right)^\tau \right)^{1/\tau} \sim \left(\sum_{Q \in D} \left(|Q|^{-\alpha/n} \omega_r(f, Q)_\tau \right)^\tau \right)^{1/\tau},$$

$$\omega_r(f, Q)_\tau := \sup_{h \in \mathbb{R}^n} \left\| \Delta_h^r(f, Q, \cdot) \right\|_{L_\tau(Q)}.$$

The following theorem generalizes what we showed for the univariate case

Theorem Let $f(x)=\mathbf{1}_{\tilde{\Omega}}(x)$, $\tilde{\Omega}\subset[0,1]^n$, a domain with smooth boundary. Then $f\in B_\tau^\alpha$, $\alpha<1/\tau$.

Proof For $\Omega=[0,1]^n$, with $l(Q)$ denoting the level of the cube Q, we may take the sum over $k\geq 0$

$$|f|_{B_\tau^\alpha}\sim\left(\sum_{k=0}^{\infty}\left(2^{k\alpha}\omega_r\left(f,2^{-k}\right)_\tau\right)^\tau\right)^{1/\tau}\sim\left(\sum_{Q\in D,l(Q)\geq 0}\left(|Q|^{-\alpha/n}\omega_r\left(f,Q\right)_\tau\right)^\tau\right)^{1/\tau}.$$

For any Q, we have that $\omega_r(f,Q)_\tau=0$, if $\partial\tilde{\Omega}\cap Q=\varnothing$. Otherwise, if $l(Q)=k$,

$$\omega_r(f,Q)_\tau\leq C\|f\|_{L_\tau(Q)}\leq C\left(\int_Q 1^\tau\right)^{1/\tau}=C|Q|^{1/\tau}=C2^{-kn/\tau}.$$

Therefore,

$$\begin{aligned}|f|_{B_\tau^\alpha}^\tau&\leq C\sum_{l(Q)\geq 0}\left(|Q|^{-\alpha/n}\omega_r\left(f,Q\right)_\tau\right)^\tau\\&\leq C\sum_{k=0}^{\infty}\left(2^{k\alpha}2^{-kn/\tau}\right)^\tau\#\left\{Q:l(Q)=k,\ Q\cap\partial\tilde{\Omega}\neq\varnothing\right\}\\&=C\sum_{k=0}^{\infty}2^{k(\alpha\tau-n)}\#\left\{Q:l(Q)=k,\ Q\cap\partial\tilde{\Omega}\neq\varnothing\right\}\end{aligned}$$

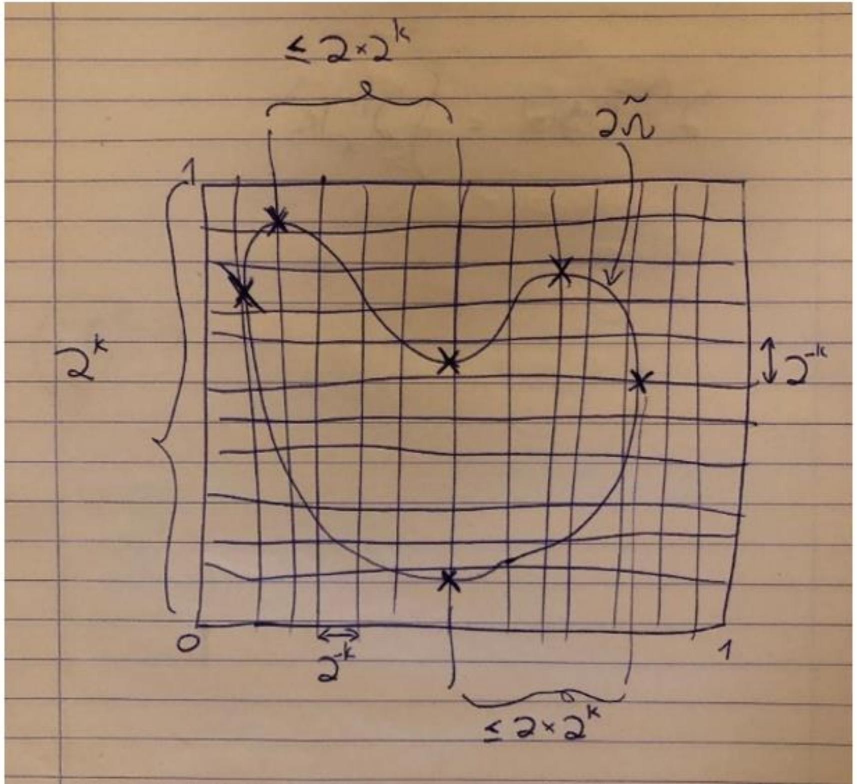
We argue that

$$\#\left\{Q : l(Q)=k,\ Q\cap\partial\tilde{\Omega}\neq\varnothing\right\}\le c\left(\tilde{\Omega}\right)2^{k(n-1)}.\ (*)$$

This implies that if $\alpha<1/\tau$

$$\left|f\right|_{B_\tau^\alpha}^{\tau}\le C\sum_{k=0}^{\infty}2^{k(\alpha\tau-n)}2^{k(n-1)}=C\sum_{k=0}^{\infty}2^{k(\alpha\tau-1)}<\infty.$$

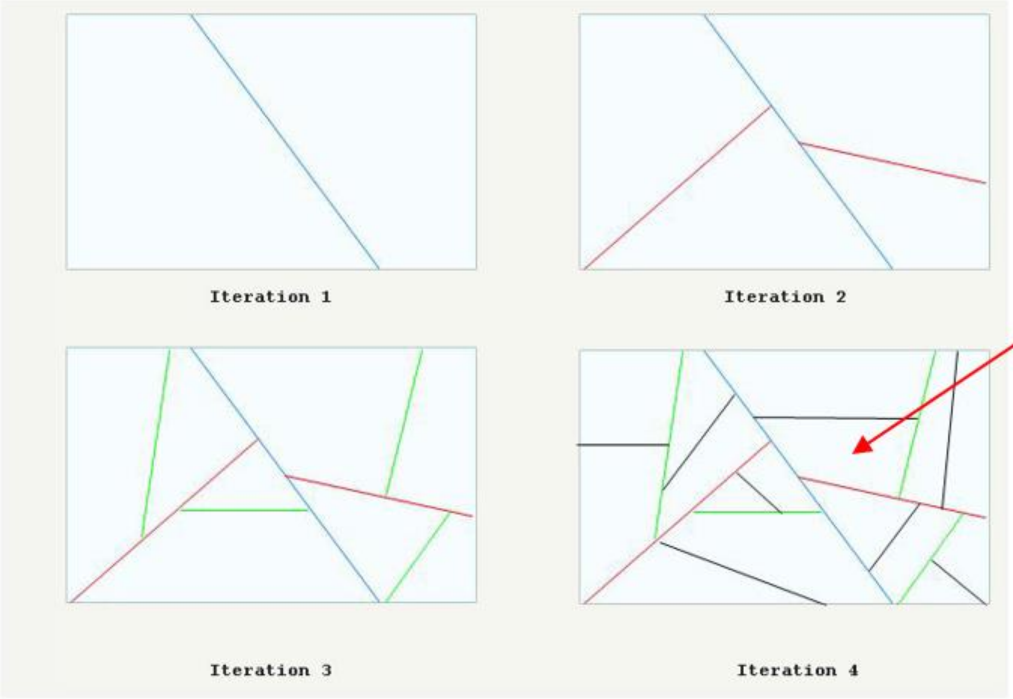
Let's get back to the estimate (*). Let use show a picture argument for $\tilde{\Omega}\subset[0,1]^2$. There is a finite number of points where the gradient of the boundary of the domain is aligned with one of the main axes. Between these points, the boundary segments are monotone in x_1 and x_2, and therefore can only intersect at most 2×2^k dyadic cubes.



The mathematical foundations of decision trees

For the theory of geometric approximation in higher dimensions we generalize to anisotropic partitions of trees over $[0,1]^n$ (replacing dyadic cubes!)

$$|f|_{B_\tau^\alpha(\mathcal{T})} := \left(\sum_{\Omega\in\mathcal{T}} \left(|\Omega|^{-\alpha} \omega_r(f,\Omega)_\tau \right)^\tau \right)^{1/\tau}$$



$\Omega \in \mathcal{T}$

Approximation Spaces

Let $\Phi := \{\Phi_N\}_{N\geq 0}$, each Φ_N is a set of functions in a (quasi) Banach space X, satisfying:

(i) $0 \in \Phi_N$, $\Phi_0 := \{0\}$,

(ii) $\Phi_N \subset \Phi_{N+1}$,

(iii) $a\Phi_N = \Phi_N$, $\forall a \neq 0$,

(iv) $\Phi_N + \Phi_N \subset \Phi_{cN}$, for some constant $c(\Phi)$,

(v) $\overline{\bigcup_N \Phi_N} = X$,

(vi) Each $f \in X$ has a near best approximation from Φ_N. That is, there exists a constant $C(\Phi)$, such that for any N, one has $\varphi_N \in \Phi_N$,

$$\|f - \varphi_N\|_X \leq C E_N(f)_X, \qquad E_N(f)_X := \inf_{\varphi \in \Phi_N} \|f - \varphi\|_X .$$

Examples for Φ_N

<u>Linear</u>

- Trigonometric polynomials of degree $\leq N$, $X = L_p\left([-\pi,\pi]^n\right)$.
- Algebraic polynomials of degree $\leq N$, $X = L_p[-1,1]$.
- Uniform dyadic knot piecewise polynomials over pieces of length 2^{-N}, of fixed order r, $X = L_p[0,1]$.
- Shift invariant refinable spaces $\Phi_N := S(\phi)^{2^{-N}}$, $S(\phi) \subset S(\phi)^{1/2}$, $X = L_p(\mathbb{R}^n)$.

<u>Nonlinear/Adaptive</u>

- Rational functions of degree $\leq N$, $X = L_p[-1,1]$,
- Free knot piecewise polynomials of fixed order r over N non-uniform intervals, $X = L_p[0,1]$.
- N-term wavelets $\Phi_N = \Sigma_N := \left\{ \sum_{\#I \leq N} c_I \psi_I \right\}$, $X = L_2(\mathbb{R}^n)$.

Def Approximation spaces for $\alpha > 0$, $0 < q \le \infty$, $f \in X$,

$$|f|_{A_q^\alpha} := \begin{cases} \left(\sum_{N=1}^{\infty} \left[N^\alpha E_N(f) \right]^q \frac{1}{N} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{N \ge 1} N^\alpha E_N(f), & q = \infty. \end{cases}$$

$$\|f\|_{A_q^\alpha} := \|f\|_X + |f|_{A_q^\alpha}.$$

One can show

$$|f|_{A_q^\alpha} \sim \begin{cases} \left(\sum_{m=0}^{\infty} \left[2^{m\alpha} E_{2^m}(f) \right]^q \right)^{1/q}, & 0 < q < \infty, \\ \sup_{m \ge 0} 2^{m\alpha} E_N(f), & q = \infty. \end{cases}$$

Goal: Fully characterize approximation spaces by smoothness spaces (iff)

Characterization of approximation spaces

1. Trigonometric polynomials

$X = L_p[-\pi,\pi]$, $1 \le p \le \infty$, Φ_N trigonometric polynomials of degree N

$$A_q^\alpha(L_p) \sim B_q^\alpha(L_p).$$

2. Dyadic univariate piecewise polynomials

$X = L_p[0,1]$, Φ_N piecewise polynomials of degree $d \ge 0$, over uniform subdivision of 2^N intervals.

For $1 \le p \le \infty$, $\alpha < r - 1 + 1/p$, $0 < q \le \infty$,

$$A_q^\alpha(L_p) \sim B_q^\alpha(L_p).$$

3. Adaptive non-uniform univariate piecewise polynomials

$$A_\tau^\alpha(L_p) \sim B_\tau^\alpha, \quad \frac{1}{\tau} = \alpha + \frac{1}{p}.$$