Function Space Theory Fall 2017:

Theorem list for the exam

General information

1. **Definition of Space of Homogeneous Type:** Let X be equipped with a quasi-distance ρ and a measure $\mu(E)=|E|$. We assume that the following hold for 'balls':
   a. $\exists c_1 > 1$ such that $\forall x, y \in X$ and $\delta > 0$, $B(x,\delta)\cap B(y,\delta)\neq\emptyset \Rightarrow B(y,\delta)\subseteq B(x,c_1\delta)$,
   b. $\exists c_2 > 1$ such that $|B(x,c_1\delta)| \le c_2 |B(x,\delta)|$, $\forall x \in X$, $\delta > 0$.

2. **Definitions of maximal functions:** Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. Recall the maximal functions:

$$M_\phi f(x) := \sup_{t>0} |f * \phi_t(x)|, \qquad \phi_t(x) := t^{-n}\phi(t^{-1}x),$$

$$M_\phi^* f(x) := \sup_{t>0} \sup_{|y-x|<t} |f * \phi_t(y)|,$$

$$M_{\phi,N}^{**} f(x) := \sup_{t>0, z\in\mathbb{R}^n} |f * \phi_t(x-z)| \left(1+\frac{|z|}{t}\right)^{-N},$$

$$M_{\mathcal{F}} f(x) := \sup_{\phi\in\mathcal{S}_{\mathcal{F}}} |M_\phi f(x)|, \qquad \mathcal{S}_{\mathcal{F}} := \left\{\phi\in\mathcal{S}(\mathbb{R}^n) : \|\phi\|_{\alpha,\beta}\le 1,\ (\alpha,\beta)\in\mathcal{F}\right\},$$

   where $\mathcal{F}$ is a finite set of Schwartz semi-norms.

Theorems for 20 points

1. [L_2 via Fourier transform] Let $f, h \in L_2(\mathbb{R})$. Prove that
   a. $\langle f, h\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{h}\rangle$,
   b. At each point $x_0 \in \mathbb{R}$ where f is continuous

$$f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w) e^{iwx_0} dw.$$

Remarks:

(i) Remember to first prove (a) for $f, h \in \mathcal{S}(\mathbb{R})$ and then use density.

(ii) You may use the formulas for Fourier transforms of Gaussians.

2. [Equivalent definitions of Schwartz class] Let $\phi \in C^\infty(\mathbb{R}^n)$. Prove that the following are equivalent
   a. For any $\alpha, \beta \in \mathbb{Z}_+^n$, there exists $C_{\alpha,\beta} > 0$ such that $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| \le C_{\alpha,\beta}$,

b. For any $N>0$ and $\beta\in\mathbb{Z}_+^n$, there exists $C_{N,\beta}>0$ such that $\left|\partial^\beta\phi(x)\right|\le C_{N,\beta}\left(1+|x|\right)^{-N}$, $\forall x\in\mathbb{R}^n$.

3. [Vitali covering Lemma] Assume X is a Space of Homogenous type. Prove that there exists a constant $c>0$, such that for any union of finite balls $E=\bigcup_{k=1}^m B_j$, one may select a subset $\left\{\tilde{B}_j\right\}_{j=1}^J$ of pairwise disjoint balls such that

$$|E|\le c\sum_{j=1}^J\left|\tilde{B}_j\right| .$$

4. [Equivalence of Maximal Functions I] Prove that for $\phi\in\mathcal{S}\left(\mathbb{R}^n\right)$, $\int\phi=1$, and $N>0$, there exist a set of semi-norms $\mathcal{F}$ and a constant $c\left(\phi,N,\mathcal{F}\right)>0$, such that for any $f\in\mathcal{S}'\left(\mathbb{R}^n\right)$, and $x\in X$,
$M_{\mathcal{F}}f(x)\le cM_{\phi,N}^{**}f(x)$.
Remark: You may use the result of Theorem 9 below.

5. [Equivalence of Besov Spaces] Prove that for any choice $r\ge\lfloor\alpha\rfloor+1$, we get the same Besov space using the semi-norm definition for $1\le p\le\infty$, $1\le q<\infty$,

$$|f|_{B_q^\alpha(L_p)}:=\left(\int_0^\infty\left(t^{-\alpha}\omega_r\left(f,t\right)_p\right)^q\frac{dt}{t}\right)^{1/q}.$$

Remark You may use the Hardy and Marchaud inequalities.

6. [Discrete Besov norm] Define the integral form of the Besov semi-norm (over $[0,1]$ or $[0,\infty)$). Prove the equivalency with the discrete dyadic form.

Theorems for 30 points

7. [Interpolation of Weak L_p spaces] Let $0<p<q\le\infty$ and let $f\in L_{p,\infty}(\Omega)\cap L_{q,\infty}(\Omega)$. Prove that $f\in L_r(\Omega)$ for any $p<r<q$.

8. [Maximal Theorem] Assume X is a Space of Homogenous type. Define for any measurable function $f:X\to\mathbb{R}$ and point $x\in X$

$$Mf(x):=\sup_{\delta>0}\frac{1}{\left|B(x,\delta)\right|}\int_{B(x,\delta)}\left|f(y)\right|dy.$$

a. Prove that if $f\in L_1(X)$, then $\|Mf\|_{1,\infty}:=\sup_{\alpha>0}\alpha\left|\left\{x\in X:\left|f(x)\right|>\alpha\right\}\right|\le c\|f\|_1$.

b. Prove that if $f \in L_p(X)$, $1 < p \le \infty$, then $\|Mf\|_p \le c\|f\|_p$.

Remarks:

(i) You may use the equality for $g \in L_p(X)$, $0 < p < \infty$,

$$\|g\|_p^p = p\int_0^\infty \left|\{x : |g(x)| > \alpha\}\right| \alpha^{p-1} d\alpha .$$

(ii) You may use Vitali's covering lemma.

9. [Representation of Schwartz functions] Let $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^n)$, $\int \Psi = \int \Phi = 1$. Prove there exists a sequence $\eta_k \in \mathcal{S}$, $k \ge 0$, such that:

a. $\Psi = \sum_{k=0}^{\infty} \eta_k * \Phi_{2^{-k}}$, where $\Phi_t(x) := t^{-n}\Phi(t^{-1}x)$, $x \in \mathbb{R}^n$.

b. The sequence $\{\eta_k\}$ decays rapidly, i.e., for any $\alpha, \beta \in \mathbb{Z}_+^n$ and $M > 0$, there exists $C_{\alpha,\beta}(M) > 0$, such that $\sup_{x \in \mathbb{R}^n} \left|x^\alpha \partial^\beta \eta_k(x)\right| \le C_{\alpha,\beta}(M) 2^{-kM}$.

10. [Equivalence of Maximal functions II] Prove that for $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\int \phi = 1$, and $0 < p < \infty$, there exists a constant $c > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\left\|M_\phi^* f\right\|_p \le c\left\|M_\phi f\right\|_p$.

Remark: You may assume $\left\|M_\phi^* f\right\|_p < \infty$.

11. [Atomic Hardy space] Prove that $H_A^p(\mathbb{R}^n) \subseteq H^p(\mathbb{R}^n)$, $0 < p \le 1$.

Remarks:

(i) First show that for appropriately chosen $\phi \in \mathcal{S}$, there exists a constant $c > 0$, such that for any p-atom a, $\left\|M_\phi a\right\|_p \le c$.

(ii) You may use the quasi-triangle inequality, i.e. for any $\{f_k\}$, $f_k \in L_p(\mathbb{R}^n)$, $k \ge 1$,

$$\left\|\sum_j f_j\right\|_p^p \le \sum_j \left\|f_j\right\|_p^p .$$

12. [Sobolev semi-norm estimates] For any $0 \le j < r$, $1 \le p < \infty$, there exists a constant $c > 0$, such that for any $f \in W_p^r(\mathbb{R}^n)$ and $\varepsilon > 0$

$$|f|_{j,p} \le c\left(\varepsilon |f|_{r,p} + \varepsilon^{-j(r-j)} \|f\|_p\right).$$

Remark: You may use the following result. For any $g \in C^2[0,\delta]$, $\delta > 0$, and $1 \le p < \infty$,

$$\left|g'(0)\right| \le \frac{c}{\delta}\left(\delta^p \int_0^\delta \left|g''(u)\right|^p du + \delta^{-p} \int_0^\delta \left|g(u)\right|^p du \right).$$

13. [Relation of modulus and Sobolev semi-norm] Prove that for $f \in W_p^r(\mathbb{R})$, $1 \le p \le \infty$,

$$\omega_r(f,t)_p \le t^r \left|f\right|_{r,p} , \qquad t > 0.$$

Remark: You may use the Minkowski inequality.

14. [Marchaud inequality] Prove that for any $1 \le k < r$ and $1 \le p \le \infty$, there exists a constant $c > 0$, such that for any $f \in L_p(\mathbb{R})$,

$$\omega_k(f,t)_p \le ct^k \int_t^\infty \frac{\omega_r(f,s)_p}{s^{k+1}} ds , \quad t > 0.$$