Mathematical Foundations of Machine Learning, Spring 2020: Assignment I

1. [Generalized Hölder inequality] Let $0 < p, p_1, \dots, p_m \le \infty$, for $m \ge 2$, and $f_k \in L_{p_k}\left(\mathbb{R}^n\right)$, $k = 1, \dots, m$. Assume that

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}.$$

Using the 'standard' Hölder inequality, show that

$$\left\| \prod_{k=1}^{m} f_k \right\|_p \le \prod_{k=1}^{m} \| f_k \|_{p_k} .$$

2. [Minkowski integral inequality] Prove that for $1 \le p < \infty$ and a measurable function $F(x,t) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$\left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \left| F(x,t) \right| dx \right)^p dt \right)^{1/p} \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} \left| F(x,t) \right|^p dt \right)^{1/p} dx .$$

Hints: for $1 < p < \infty$ $\left(\int_{\mathbb{R}^n} \left| F(x,t) \right| dx \right)^p = \left(\int_{\mathbb{R}^n} \left| F(x,t) \right| dx \right) \left(\int_{\mathbb{R}^n} \left| F(y,t) \right| dy \right)^{p-1}$, change order of integration of t and x, use Hölder.

3. Let $f(x) := \sum_{m=1}^{M} c_m \mathbf{1}_{[2m,2m+1]}(x)$. Compute the modulus $\omega_1(f,t)_p$, for all $0 < t < 1/2$, and $0 < p \le \infty$.

4. Prove the following equality for any $N \ge 1$, $x, h \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta_{Nh}^r(f,x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta_h^r\left(f, x + k_1 h + \ldots + k_r h\right).$$

Hint: recall we proved in class for $r = 1$. Now apply induction on r. <u>Make sure the notations are correct</u>.

5. Recall that a function $g \in L_1\left(\mathbb{R}^n\right)$ is the **distributional derivative** of $f \in L_1\left(\mathbb{R}^n\right)$, $g := \partial^\alpha f$, $\alpha \in \mathbb{Z}_+^n$, if

$$\int_{\mathbb{R}^n} g\phi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f \partial^\alpha \phi , \qquad \forall \phi \in C_0^\infty\left(\mathbb{R}^n\right) .$$

Prove $H'(x) = \begin{cases} 1, & -1 \le x < 0, \\ -1, & 0 \le x \le 1, \\ 0, & \text{else.} \end{cases}$ where $H(x) := \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & \text{else.} \end{cases}$