Introduction to function spaces: Assignment III

1. Prove the following equality for any $N \geq 1$, $x, h \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta^r_{Nh}(f, x) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_r=0}^{N-1} \Delta^r_h(f, x + k_1 h + \ldots + k_r h).$$

Hint: recall we proved in class for $r = 1$. Now apply induction on r. <u>Make sure the notations are correct</u>.

2. Let $f(x) := \sum_{m=1}^{M} c_m \mathbf{1}_{[2m,2m+1]}(x)$. Compute the modulus $\omega_1(f, t)_p$, for all $0 < t < 1/2$, and $0 < p \leq \infty$.

3. Recall that we proved in class for $g \in C^r(\mathbb{R}) \cap W_p^r(\mathbb{R})$, $1 \leq p \leq \infty$, that

$$\omega_r(g, t)_{L_p(\Omega)} \leq C(r, n) t^r |g|_{W_p^r(\Omega)}, \qquad \forall t > 0.$$

Complete the proof for a general $g \in W_p^r(\mathbb{R})$, $1 \leq p < \infty$ by using a 'density' argument: there exists a sequence of functions $\{g_k\} \subset C^r(\mathbb{R}) \cap W_p^r(\mathbb{R})$, such that $\|g_k - g\|_{W_p^r(\mathbb{R})} \underset{k \to \infty}{\to} 0$.

4. [Minkowski integral inequality] Prove that for $1 \leq p < \infty$ and a measurable function $F(x,t) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$\left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |F(x,t)| \, dx \right)^p dt \right)^{1/p} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |F(x,t)|^p \, dt \right)^{1/p} dx.$$

Hints: for $1 < p < \infty$ $\left(\int_{\mathbb{R}^n} |F(x,t)| \, dx \right)^p = \left(\int_{\mathbb{R}^n} |F(x,t)| \, dx \right) \left(\int_{\mathbb{R}^n} |F(y,t)| \, dy \right)^{p-1}$, change order of integration of t and x, use Hölder.