Foundations of Approximation Theory - Local Polynomial Approximation

Shai Dekel

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Contents

CHAPTER 1

Local polynomial approximation over convex domains in $\mathbb{R}^n$

In this chapter we review the theory of local approximation using multivariate algebraic polynomials of relatively low-degree over 'regular' domains in $\mathbb{R}^n$. By 'regular' domains we mean domains which have nice geometric properties as we define precisely in the next section. This local smoothness analysis and approximation by algebraic polynomials is the critical component that allows us to construct anisotropic spaces that are a 'true' generalization of the classical isotropic function spaces over $\mathbb{R}^n$. This is in contrast to general spaces of homogeneous type that do not have enough 'structure' and thus function spaces defined over them are limited in various ways. In section 1.2 we review the analysis tools we use to quantity local function smoothness. In section 1.3 we provide some properties of algebraic polynomials over convex domains. We then proceed to provide estimates for the degree of polynomial approximation over domains, where section 1.4 is focused on approximation in the p-norm, with $1 \leq p \leq \infty$, of the Sobolev class and section 1.5 is mostly dedicated to approximation in the p-norm, with $0 < p < 1$.

1.1. Geometric properties of regular bounded domains

Definition 1.1. We denote by $B(x_0, r)$ the Euclidean ball in $\mathbb{R}^n$ with center at $x_0 \in \mathbb{R}^n$ and radius $r > 0$. The image of the Euclidean unit ball $B^* := B(0,1)$ via an affine transform will be called an ***ellipsoid***. For a given ellipsoid θ we let A_θ be an affine transform such that $\theta = A_\theta(B^*)$. Denoting by $v_\theta := A_\theta(0)$ the 'center' of θ we have

$$A_\theta(x) = M_\theta x + v_\theta, \tag{1.1}$$

where M_θ is a positive definite $n \times n$ matrix.

Any positive definite $n \times n$ real matrix M can be represented in the form $M = UDU^{-1}$, where the matrix U is $n \times n$ orthogonal matrix and D is diagonal and $D = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. It is easy to see that $\sigma_1^2 \geq \cdots \geq \sigma_n^2$ are the eigenvalues of $M^T M$ and $\sigma_1^{-2} \leq \cdots \leq \sigma_n^{-2}$ are the eigenvalues of $(M^{-1})^T M^{-1}$. Hence

$$\|M\|_{\ell_2 \to \ell_2} = \sigma_1 \quad \text{and} \quad \|M^{-1}\|_{\ell_2 \to \ell_2} = 1/\sigma_n. \tag{1.2}$$

These norms have a clear geometric meaning. Thus if M_θ is as in (1.1), then $\operatorname{diam}\theta = 2\|M_\theta\|_{\ell_2 \to \ell_2} = 2\sigma_1$. One can also say that the width of θ is $2\sigma_n$, since σ_n is the length of the smallest axis of θ. The ellipsoid is in fact the prototypical example of bounded convex domains.

Definition 1.2. A set $\Omega \subseteq \mathbb{R}^n$ is ***convex*** if for any two points $x, y \in \Omega$ the line segment $[x, y]$ is contained in Ω. The ***convex-hull*** of a set $A \subset \mathbb{R}^n$ is the 'minimal' convex set containing A, which is given by the intersection of all convex sets containing A.

PROPOSITION 1.3. **John's Theorem** [**21**] For any bounded convex domain $\Omega \subset \mathbb{R}^n$ exists an ellipsoid $\theta \subseteq \Omega$ such that if v_θ is the center of θ, then

$$\theta = v_\theta + M_\theta(B^*) \subseteq \Omega \subseteq v_\theta + n(M_\theta(B^*)).$$

This implies that the affine transform $A_\theta^{-1}(x) := M_\theta^{-1}(x - v_\theta)$ gives

$$B(0,1) \subseteq A_\theta^{-1}(\Omega) \subseteq B(0,n). \tag{1.3}$$

It is interesting to note that John's ellipsoid θ, is the ellipsoid with maximal volume such that $\theta \subseteq \Omega$. In some sense this means that θ 'covers' Ω sufficiently well. Our approximation theoretical applications of John's theorem utilize the fact that bounded convex domains are essentially equivalent to the Euclidean ball B^* up to an affine transformation and scale n.

DEFINITION 1.4. A domain $\Omega \subset \mathbb{R}^n$ is ***star-shaped*** with respect to a Euclidean ball B (or a point x_0), if for any point $x \in \Omega$, the convex-hull of $\{x\} \cup B$ (or the line segment $[x, x_0]$) is contained in Ω.

We call the set

$$V := \{x \in \mathbb{R}^n : \quad x = 0 \vee 0 < |x| \leq \rho, \angle(x,v) \leq \kappa/2\},$$

a **finite cone** of axis direction v, height ρ, and aperture angle κ, where $\angle(x,v)$ is the angle between x and v. For $z \in \mathbb{R}^n$, the set $z+V := \{z+y, y \in V\}$ is a translate of V, which is a finite cone with head vertex at z. A cone V' is **congruent** to V, if it can be obtained from V thorough a rigid motion.

We now define notions of 'minimally smooth' domains (see pages 81-83 in [**1**], page 189 in [**28**]). Although we will be mostly dealing with bounded convex domains and, in particular the special case ellipsoids, some of the results we use or prove hold for more general types of domains.

DEFINITION 1.5. A domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the **uniform cone property** if there exist numbers $\delta > 0$, $L > 0$, a finite cover of open sets $\{U_j\}_{j=1}^J$ of $\partial\Omega$, and a corresponding collection $\{V_j\}_{j=1}^J$ of finite cones, each congruent to some fixed cone V, such that

(i) $\mathrm{diam}(U_j) \leq L$, $1 \leq j \leq J$.
(ii) For any $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) < \delta$, we have $x \in \bigcup_{j=1}^J U_j$.
(iii) If $x \in \Omega \cap U_j$, then $x + V_j \subseteq \Omega$, $1 \leq j \leq J$.

We will say the domain satisfies the **overlapping uniform cone property** if in addition the following condition is satisfied

(iv) For every pair of points $x_1, x_2 \in \Omega$, such that $|x_1 - x_2| < \delta$ and $\mathrm{dist}(x_i, \partial\Omega) < \delta$, $i = 1,2$, there exists an index j such that $x_i \in U_j$, $i = 1,2$

THEOREM 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain such that $B(0,R_1) \subseteq \Omega \subseteq B(0,R_2)$, for some fixed $0 < R_1 < R_2$. Then Ω satisfies the overlapping uniform cone property with parameters that depend only on n, R_1 and R_2.*

1.2. The Modulus of smoothness

From this point, we assume that domains $\Omega \subset \mathbb{R}^n$ are measurable with a nonempty interior and that all functions are measurable and real.

1.2.1. Definitions And Basic Properties.

Definition 1.7. Let $W_p^r(\Omega)$, $1 \le p \le \infty$ $r \in \mathbb{N}$ denote the ***Sobolev spaces***, namely, the spaces of functions $g \in L_p(\Omega)$, which have all their distributional derivatives of order up to r, in $L_p(\Omega)$. The norm of the Sobolev space is given by

$$\|g\|_{W_p^r(\Omega)} := \|g\|_{r,p} = \sum_{|\alpha| \le r} \|\partial^\alpha g\|_{L_p(\Omega)}, \tag{1.4}$$

where for $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$, while the semi-norm is given by

$$|g|_{W_p^r(\Omega)} := |g|_{r,p} = \sum_{|\alpha| = r} \|\partial^\alpha g\|_{L_p(\Omega)}. \tag{1.5}$$

One can show [**1**] that the norms of the derivatives of order $1 \le j < r$ can be bounded using the p-norm and the derivatives of order r. Hence

$$\|g\|_{W_p^r(\Omega)} \sim \|g\|_{L_p(\Omega)} + |g|_{W_p^r(\Omega)}. \tag{1.6}$$

Definition 1.8. The ***K-functional of order r*** of $f \in L_p(\Omega)$, $1 \le p \le \infty$, (see e.g. [**17**]) is defined by

$$K_r(f,t)_p := K(f,t,L_p(\Omega),W_p^r(\Omega)) := \inf_{g \in W_p^r(\Omega)} \{\|f-g\|_p + t|g|_{r,p}\}, \qquad t > 0. \tag{1.7}$$

For a bounded domain Ω, we denote

$$K_r(f,\Omega)_p := K(f, \operatorname{diam}(\Omega)^r)_p. \tag{1.8}$$

It is important to note that the K-functional is unsuitable as a measure of smoothness if $0 < p < 1$ (see [**18**]). For $f \in L_p(\Omega)$, $0 < p \le \infty$, $h \in \mathbb{R}^d$, and $r \in \mathbb{N}$, we define the rth order **difference operator** $\Delta_h^r f : \Omega \to \mathbb{R}$, by

$$\Delta_h^r(f,x) := \Delta_h^r(f,\Omega,x) := \begin{cases} \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x+kh), & [x, x+rh] \subset \Omega, \\ 0, & \text{otherwise}, \end{cases} \tag{1.9}$$

where $[x,y]$ denotes the line segment connecting any two points $x, y \in \mathbb{R}^n$.

Definition 1.9. The ***modulus of smoothness of order r*** is defined by

$$\omega_r(f,t)_p = \omega_r(f,\Omega,t)_p := \sup_{|h| \le t} \left\|\Delta_h^r(f,\Omega,\cdot)\right\|_{L_p(\Omega)}, \qquad t > 0, \tag{1.10}$$

where for a vector $h \in \mathbb{R}^n$, $|h|$ denotes the l_2-norm of h. For a bounded domain Ω we also denote

$$\omega_r(f,\Omega)_p := \omega_r(f, \operatorname{diam}(\Omega))_p. \tag{1.11}$$

We list some of the properties of the modulus of smoothness that we shall use throughout the book (see [**17**]) for more details),

Proposition 1.10. Let $\Omega \subseteq \mathbb{R}^n$ and $f, g \in L_p(\Omega)$, $0 < p \le \infty$. Then for any $t > 0$,

(i) $\omega_r(f,t)_p \le c(r,p)\|f\|_p$. In more general form, for any $0 \le k < r$, $\omega_r(f,t)_p \le C(r,k,p)\omega_k(f,t)_p$, (where $\omega_0(f,\cdot)_p = \|f\|_p$).
(ii) $\omega_r(f+g,t)_p \le c(p)(\omega_r(f,t)_p + \omega_r(g,t)_p)$.
(iii) For any $\lambda \ge 1$, $\omega_r(f,\lambda t)_p \le (\lambda+1)^r \omega_r(f,t)_p$, for $1 \le p \le \infty$, and $\omega_r(f,\lambda t)_p^p \le (\lambda+1)^r \omega_r(f,t)_p^p$, for $0 < p < 1$.

(iv) If $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$, then for any vector $h \in \mathbb{R}^n$, and domain Ω,

$$\left\|\Delta_h^r(f,\Omega_1,\cdot)\right\|_{L_p(\Omega)} \le \left\|\Delta_h^r(f,\Omega_2,\cdot)\right\|_{L_p(\Omega)}, \tag{1.12}$$

and

$$\omega_r(f,\Omega_1,t)_p \le \omega_r(f,\Omega_2,t)_p.$$

1.2.2. Relations Between The Modulus Of Smoothness And K-Functional. We now present the relationship of the difference and the derivative operators using B-splines. We recall the univariate B-spline of order 1 (degree 0), $N_1(u) := 1_{[0,1]}(u)$. Then, the B-spline of order r (degree $r-1$), is defined by $N_r := N_{r-1} * N_1$. The B-spline of order r is supported on $[0,r]$, is in C^{r-1} and is a piecewise polynomial of degree $r-1$ over the integer intervals. For $h_1 > 0$, we define $N_r(u,h_1) := h_1^{-1}N_r(h_1^{-1}u)$. Let $g \in C^r(\Omega)$, and let $h \in \mathbb{R}^n$, with $|h| = h_1 > 0$, then if the segment $[x,x+h]$ is contained in Ω, we have for $\xi := h_1^{-1}h$, $G(u) := g(x+u\xi)$, $u \in \mathbb{R}$,

$$\begin{aligned} h_1^{-1}\Delta_h(g,x) &= h_1^{-1}\int_0^{h_1} G'(u)du \\ &= \int_{\mathbb{R}} G'(u)N_1(u,h_1)du \\ &= \int_{\mathbb{R}} D_\xi g(x+u\xi)N_1(u,h_1)du, \end{aligned}$$

where

$$D_\xi g(y) := \lim_{u\to 0}\frac{g(y+u\xi)-g(y)}{u}.$$

By induction, we get for $r \ge 1$

$$h_1^{-r}\Delta_h^r(g,x) = \int_{\mathbb{R}} G^{(r)}(u)N_r(u,h_1)du = \int_{\mathbb{R}} D_\xi^r g(x+u\xi)N_r(u,h_1)du. \tag{1.13}$$

Based on the relation (1.13) we can bound the modulus of smoothness of the Sobolev class.

THEOREM 1.11. *For $g \in W_p^r(\Omega)$, $r \ge 1$, $1 \le p \le \infty$,*

$$\omega_r(g,t)_p \le c(n,r)t^r|g|_{r,p}, \quad t > 0. \tag{1.14}$$

PROOF. Let $g \in C^r(\Omega)$. Since $D_\xi g = \sum_{i=1}^n \xi_i \frac{\partial g}{\partial x_i}$, and $|\xi| = 1$, we have that $\|D_\xi g\|_p \le |g|_{1,p}$. One can see that by induction, $D_\xi^r g = \sum_{|\alpha|=r} c_\alpha D^\alpha g$, with $|c_\alpha| \le c(n,r)$. This implies that $\|D_\xi^r g\|_p \le c(n,r)|g|_{r,p}$. Let $h \in \mathbb{R}^n$, with $0 < |h| = h_1 \le t$, $\xi := h_1^{-1}h$ and denote $\Omega_{r,h} := \{x \in \Omega : \quad [x,x+rh] \subset \Omega\}$ Applying (1.9), (1.13) and then Minkowski's inequality for $1 \le p \le \infty$, yields

$$\begin{aligned} \|\Delta_h^r(g,\cdot)\|_{L_p(\Omega)} &= \|\Delta_h^r(g,\cdot)\|_{L_p(\Omega_{h,r})} \\ &\le t^r\Big\|\int_{\mathbb{R}} D_\xi^r g(\cdot+u\xi)N_r(u,h_1)du\Big\|_{L_p(\Omega_{h,r})} \\ &\le t^r\left\|D_\xi^r g\right\|_{L_p(\Omega)} \\ &\le c(n,r)t^r|g|_{r,p}. \end{aligned}$$

Taking supremum over all $h \in \mathbb{R}^n$, $|h| \le t$, gives (1.14) for functions in $C^r(\Omega)$. For $1 \le p < \infty$ we apply a standard density argument to obtain (1.14) for the Sobolev class. □

PROPOSITION 1.12. [**22**] Let $\Omega \subset \mathbb{R}^n$ satisfy the Uniform Cone property (see Definition 1.5) and let $1 \le p \le \infty$ and $r \ge 1$. Then there exist constants $C_1(\Omega, p, n, r), C_2(n, r) > 0$, such that for any any $f \in L_p(\Omega)$,

$$C_1 K_r(f, t^r)_p \le \omega_r(f, t)_p \le C_2 K_r(f, t^r)_p, \qquad 0 < t \le \operatorname{diam}(\Omega). \tag{1.15}$$

PROOF. To see the right hand side of (1.15), let g be any function in $W_p^r(\Omega)$. we apply (1.14) to obtain

$$\begin{aligned} \omega_r(f, t)_p &\le \omega_r(f - g, t)_p + \omega_r(g, t)_p \\ &\le 2^r \|f - g\|_p + C(n, r) t^r |g|_{r,p} \\ &\le C(n, r)(\|f - g\|_p + t^r |g|_{r,p}). \end{aligned}$$

Therefore, by taking the infimum over all such $g \in W_p^r(\Omega)$, we obtain the right hand side of (1.15). The left hand side is the main result of [**22**]. We note that the uniform cone property is a slightly stronger assumption then what is used in [**22**]. □

Note that, while C_2 in (1.15) depends only on n and r, the constant C_1 may further depend on the geometry of Ω (e.g. the parameters of the Uniform Cone properties). One can obtain a more specific left-hand side inequalities for convex domains. A first result for convex domains is

COROLLARY 1.13. Let $\Omega \subset \mathbb{R}^n$ be a convex domain such that $B(0, R_1) \subseteq \Omega \subseteq B(0, R_2)$, for some fixed $0 < R_1 < R_2$. Then for $f \in L_p(\Omega)$, $1 \le p \le \infty$, $r \ge 1$, and $0 < t \le 2R_2$,

$$C_1(r, p, R_1, R_2) K_r(f, t^r)_{L_p(\Omega)} \le \omega_r(f, t)_{L_p(\Omega)} \le C_2(n, r) K_r(f, t^r)_{L_p(\Omega)}. \tag{1.16}$$

PROOF. The right hand side of (1.16) holds by (1.15) for more general domains. To prove the left hand side inequality one applies Theorem 1.6. □

The second result on the relationship between the modulus of smoothness and the K-functional over convex domains requires using the 'local' polynomial approximation results of the next Chapter. We state it here

PROPOSITION 1.14. [**10**] Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for any $f \in L_p(\Omega)$, $1 \le p \le \infty$, $r \ge 1$,

$$K_r(f, t^r) \le C(n, r, p) \left(\left(1 - \frac{t^r}{\operatorname{diam}(\Omega)^r} \right) \mu(\Omega, t)^{-(r-1+1/p)} + 1) \right) \omega_r(f, t)_p,$$

where

$$\mu(\Omega, t) := \min_{x \in \Omega} \frac{|B(x, t) \cap \Omega|}{|B(x, t)|}, \qquad 0 < t \le \operatorname{diam}(\Omega).$$

1.3. Algebraic polynomials over domains

Let $\Pi_{r-1} := \Pi_{r-1} = \Pi_{r-1}(\mathbb{R}^n)$ denote the multivariate polynomials of total degree $r - 1$ (order r) in n variables. This is the collection of functions of the type $P(x) = \sum_{|\alpha| < r} c_\alpha x^\alpha$, where for $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$, and for $x \in \mathbb{R}^n$, $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$.

LEMMA 1.15. *Let $P \in \Pi_{r-1}$ and let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, be bounded convex domains such that $\Omega_1 \subseteq \Omega_2$ and $|\Omega_2| \le \rho |\Omega_1|$ for some $\rho > 1$. Then for $0 < p \le \infty$*

$$\|P\|_{L_p(\Omega_2)} \le C(n, r, p, \rho) \|P\|_{L_p(\Omega_1)}.$$

PROOF. Let $Ax = Mx + b$ be the affine transform for which (1.3) holds for Ω_1. Since $A^{-1}(\Omega_1) \subseteq B(0,n)$ we have

$$
\begin{aligned}
|A^{-1}(\Omega_2)| &= |A^{-1}(\Omega_1)| \frac{|A^{-1}(\Omega_2)|}{|A^{-1}(\Omega_1)|} \\
&\leq |B(0,n)|\rho := C(n,\rho).
\end{aligned} \tag{1.17}
$$

Observe that $A^{-1}(\Omega_2)$ is a convex domain that contains $A^{-1}(\Omega_1)$ and therefore also contains $B(0,1)$. Together with (1.17) this implies that the diameter of $A^{-1}(\Omega_2)$ must be bounded by a constant that depends on n and ρ, i.e., $A^{-1}(\Omega_2) \subseteq B(0,R)$, $R := R(n,\rho)$. Hence applying the equivalence of finite dimensional (quasi) normed spaces we obtain

$$
\begin{aligned}
\|P\|_{L_p(\Omega_2)} &= |\det M|^{1/p} \big\|P\big\|_{L_p(A^{-1}(\Omega_2))} \\
&\leq |\det M|^{1/p} \big\|P\big\|_{L_p(B(0,R))} \\
&\leq C|\det M|^{1/p} \big\|P\big\|_{L_p(B(0,1))} \\
&\leq C|\det M|^{1/p} \big\|P\big\|_{L_p(A^{-1}(\Omega_1))} \\
&= C\big\|P\big\|_{L_p(\Omega_1)}.
\end{aligned}
$$

□

LEMMA 1.16. *For any bounded convex domain $\Omega \subset \mathbb{R}^n$, $P \in \Pi_{r-1}$, and $0 < p,q \leq \infty$, we have*

$$
\|P\|_{L_q(\Omega)} \sim |\Omega|^{1/q-1/p} \|P\|_{L_p(\Omega)}, \tag{1.18}
$$

with constants of equivalency depending only on n, r, p, and q.

PROOF. Let $Ax = Mx + b$ be the affine transform for which (1.3) holds. Since $A(B(0,1)) = \theta$, we get from the properties of John's ellipsoid, $|\det M| \sim |\Omega|$, with constants of equivalency depending only on n. Also, by the equivalence of finite dimensional (quasi) normed spaces, for any polynomial $\tilde{P} \in \Pi_{r-1}$ we have that $\|\tilde{P}\|_{L_p(B(0,1))} \sim \|\tilde{P}\|_{L_q(B(0,n))}$ with constants of equivalency that depend only on n, r, p, and q. Let $P \in \Pi_{r-1}$, and denote $\tilde{P} := P(A\cdot)$. Then

$$
\begin{aligned}
\|P\|_{L_q(\Omega)} &= |\det M|^{1/q} \|\tilde{P}\|_{L_q(A^{-1}(\Omega))} \\
&\leq |\det M|^{1/q} \big\|\tilde{P}\big\|_{L_q(B(0,n))} \\
&\leq C|\det M|^{1/q} \big\|\tilde{P}\big\|_{L_p(B(0,1))} \\
&\leq C|\det M|^{1/q} \big\|\tilde{P}\big\|_{L_p(A^{-1}(\Omega))} \\
&\leq C|\det M|^{1/q-1/p} \big\|P\big\|_{L_p(\Omega)} \\
&\leq C|\Omega|^{1/q-1/p} \big\|P\big\|_{L_p(\Omega)}.
\end{aligned}
$$

□

PROPOSITION 1.17. [**26**] Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Then, for $1 \leq p \leq \infty$, any polynomial $P \in \Pi_{r-1}$, and $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := \sum_{i=1}^n \alpha_i \leq r-1$,

$$
\|\partial^\alpha P\|_{L_p(\Omega)} \leq C\,(n, |\alpha|)\, \mathrm{width}\,(\Omega)^{-|\alpha|}\, \|P\|_{L_p(\Omega)}\,, \tag{1.19}
$$

where width (Ω) is the diameter of the largest n-dimensional Euclidean ball that is contained in Ω.

THEOREM 1.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $1 \le p < \infty$. Then, for any $P \in \Pi_{r-1}$, we have that $\omega_r(P,t)_p = 0$, $0 < t \le \operatorname{diam}(\Omega)$. In the other direction, if Ω is also open and connected and $f \in L_p(\Omega)$, such that $\omega_r(f,\Omega)_p = 0$ for some $r \ge 1$, then there exists a polynomial $P \in \Pi_{r-1}$ such that $f = P$ a.e. on Ω.*

PROOF. The first part is a direct application of the identity (1.13), since it implies that $\Delta_h^r(P,x) = 0$, for any $x \in \Omega$, $h \in \mathbb{R}^n$. To prove the second part, we apply the Whitney decomposition of Ω into interior disjoint cubes (see e.g. the appendix in [**20**]). Namely, there exist a family of closed cubes $\{Q_k\}_k$, such that:

(i) $\cup_k Q_k = \Omega$, and the cubes Q_k, have disjoint interiors,
(ii) $\sqrt{n}l(Q_k) \le \operatorname{dist}(Q_k, \Omega^c) \le 4\sqrt{n}l(Q_k)$, where $l(Q_k)$ is the side length of Q_k,
(iii) If the boundaries of Q_k and Q_j touch, then
$$\frac{1}{4} \le \frac{l(Q_j)}{l(Q_k)} \le 4,$$
(iv) For any Q_k, there are at most 12^n cubes Q_j that touch it.

We now construct from the Whitney decomposition a cover of 'substantially' overlapping cubes $\{\tilde{Q}_k\}_k$, simply by extending the lengths of the cubes symmetrically, such that $l(\tilde{Q}_k) = 2l(Q_k)$, $k \ge 1$. By property (ii) of the Whitney decomposition, we know that each $\tilde{Q}_k$ is contained in Ω and thus $\cup_k \tilde{Q}_k = \Omega$. Also, for touching cubes Q_k and Q_j, the extensions have a 'substantial' intersection, i.e.
$$|\tilde{Q}_k \cap \tilde{Q}_j| \ge \min\{l(Q_k)/2, l(Q_j)/2\}^n.$$

Note that there is a simpler proof for the case $1 \le p < \infty$, since we can apply the machinery of the K-functional and Sobolev spaces. Indeed, the equivalence (1.15) on each extended cube $\tilde{Q}_k$, gives
$$K_r(f,\tilde{Q}_k)_p \le C\omega_r(f,\tilde{Q}_k)_p \le C\omega_r(f,\Omega)_p = 0.$$
Thus, there exists a sequence $\{g_j\}_j$, $g_j \in W_p^r(\tilde{Q}_k)$, such that $\|f - g_j\|_{L_p(\tilde{Q}_k)} + \operatorname{diam}(\tilde{Q}_k)^r |g_j|_{W_p^r(\tilde{Q}_k)} \to 0$, as $j \to \infty$. Using (1.6), we obtain that $\{g_j\}_j$ is Cauchy sequence in $W_p^r(\tilde{Q}_k)$ and so it converges to $g \in W_p^r(\tilde{Q}_k)$, with $g = f$ a.e and $|g|_{r,p} = 0$ on $\tilde{Q}_k$. We now apply on each cube $\tilde{Q}_k$ the Bramble-Hilbert Lemma below (1.36) to conclude that $f = g = P_k$ a.e., for some $P_k \in \Pi_{r-1}$, on $\tilde{Q}_k$, $\forall k$. Since Ω is a connected domain, using the 'substantial' intersections of the extended cubes of touching cubes yields that there exists a unique $P \in \Pi_{r-1}$, such that $P = P_k$, $\forall k$, which concludes the proof for $1 \le p < \infty$. □

1.4. The Bramble-Hilbert Lemma for convex domains

Given a bounded regular domain $\Omega \subset \mathbb{R}^n$, our goal is to estimate the degree of approximation of a function $f \in L_p(\Omega)$, $0 < p \le \infty$, by algebraic polynomials of total degree $r - 1$,
$$E_{r-1}(f,\Omega)_p := \inf_{P \in \Pi_{r-1}} \|f - P\|_{L_p(\Omega)}.$$

For a star-shaped domain Ω (see Definition 1.4), we denote $\rho_{max} := \max\{\rho \mid \Omega$ is star-shaped with respect to a ball $B \subseteq \Omega$ of radius $\rho\}$. The ***chunkiness parameter*** of Ω [**3**] is defined by

$$\gamma := \frac{\operatorname{diam}(\Omega)}{\rho_{max}}. \tag{1.20}$$

Note that the chunkiness parameter γ becomes larger in cases where the domain is longer and thinner. This leads to the following formulation of the Bramble-Hilbert lemma,

THEOREM 1.19. ***Bramble-Hilbert Lemma for star-shaped domains*** *Let Ω be a bounded star-shaped with respect to some ball B, with chunkiness parameter γ, and let $g \in W_p^r(\Omega)$, $1 \le p \le \infty$, $r \ge 1$. Then there exists a polynomial $P \in \Pi_{r-1}$ for which*

$$|g - P|_{k,p} \le C(n,r)(1+\gamma)^n \operatorname{diam}(\Omega)^{r-k} |g|_{r,p}, \quad k = 0, 1 \dots, r-1. \tag{1.21}$$

Before we proceed with the proof of Theorem 1.19, we require some preparation. Let $g \in C^r(\Omega)$ and recall that the classical **Taylor polynomial** of order r (degree $r-1$), at $x \in \Omega$, about a point $y \in B$, is given by

$$T_y^r g(x) := \sum_{|\alpha|<r} \frac{D^\alpha g(y)}{\alpha!}(x-y)^\alpha, \tag{1.22}$$

where $\alpha! := \prod_{i=1}^n \alpha_i!$. The **Taylor remainder** of order r is then given by

$$TR_y^r g(x) := r \sum_{|\alpha|=r} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 s^{r-1} D^\alpha g(x + s(y-x))ds, \tag{1.23}$$

which is meaningful since the segment $[x, y]$ is contained in Ω. We then have,

$$g(x) = T_y^r g(x) + TR_y^r g(x), \quad x \in \Omega.$$

Our construction of an approximating polynomial relies on averaging the Taylor polynomials over the ball B. It can be shown that there exists a cut-off function $\phi \in C^\infty$, for $B(0,1)$, with the properties:

(i) $\int_{\mathbb{R}^n} \phi(x)dx = 1$,
(ii) $\operatorname{supp}(\phi) = B(0,1)$,
(iii) $\|\phi\|_\infty \le 1$.

For any ball $B(x_0, \rho)$, the cut-off function $\phi_B := \rho^{-n}\phi(\rho^{-1}(\cdot - x_0))$, satisfies:

(i) $\int_{\mathbb{R}^n} \phi_B(x)dx = 1$,
(ii) $\operatorname{supp}(\phi_B) = B(x_0, \rho)$,
(iii) $\|\phi_B\|_\infty \le \rho^{-n}$.

The **Averaged Taylor polynomial** of $g \in C^r(\Omega)$, over $B \subseteq \Omega$, of order r (degree $r-1$), is given by

$$T_B^r g(x) := \int_B T_y^r g(x) \phi_B(y) dy, \quad x \in \Omega. \tag{1.24}$$

We also denote the **Averaged Taylor remainder** by

$$R_B^r g(x) := g(x) - T_B^r g(x).$$

LEMMA 1.20. *For $x \in \Omega$, where Ω is star-shaped with respect to $B(x_0, \rho) \subset \Omega$, and $g \in C^r(\Omega)$,*

$$R_B^r g(x) = r \sum_{|\alpha|=r} \int_{V(x)} K_\alpha(x,z) D^\alpha g(z) dz, \tag{1.25}$$

where $V(x)$ is the convex closure of $\{x\} \cup B$, and $K_\alpha = \frac{1}{\alpha!}(x-z)^\alpha K(x,z)$, with

$$|K(x,z)| \le C\,(\gamma+1)^n\,|x-z|^{-n}, \quad \gamma = \frac{\text{diam}(\Omega)}{\rho}. \tag{1.26}$$

PROOF. We fix $x \in \Omega$, and observe that by properties (i),(ii) of ϕ_B,

$$\begin{aligned} R_B^r g(x) &= g(x) - T_B^r g(x) \\ &= \int_B (g(x) - T_y^r g(x)) \phi_B(y) dy \\ &= \int_B (T R_y^r g(x)) \phi_B(y) dy \\ &= r \sum_{|\alpha|=r} \int_B \frac{(x-y)^\alpha}{\alpha!} \phi_B(y) \int_0^1 s^{r-1} D^\alpha g(x + s(y-x)) ds dy. \end{aligned}$$

We now make the change of variables (y,s) to (z,s) with $z = x + s(y-x)$, and define the integration domain

$$A := \{(z,s): \quad s \in [0,1], \quad |s^{-1}(z-x) + x - x_0| \le \rho\},$$

to obtain

$$\begin{aligned} R_B^r g(x) &= r \sum_{|\alpha|=r} \frac{1}{\alpha!} \int_A (x-z)^\alpha \phi_B(s^{-1}(z-x)+x) D^\alpha g(z) s^{-n-1} dz ds \\ &= r \sum_{|\alpha|=r} \int_{V(x)} D^\alpha g(z) \frac{1}{\alpha!} (x-z)^\alpha \int_0^1 1_A(z,s) \phi_B(s^{-1}(z-x)+x) s^{-n-1} ds dz \\ &= r \sum_{|\alpha|=r} \int_{V(x)} D^\alpha g(z) K_\alpha(x,z) dz, \end{aligned}$$

where

$$K_\alpha(x,z) := \frac{1}{\alpha!}(x-z)^\alpha K(x,z), \quad K(x,z) := \int_0^1 1_A(z,s) \phi_B(s^{-1}(z-x)+x) s^{-n-1} ds.$$

We now prove the estimate (1.26). Observe that

$$(z,s) \in A \Rightarrow \frac{|z-x|}{|x-x_0| + \rho} < s.$$

So with $t := |z-x|/(|x-x_0|+\rho)$ and property (iii) of ϕ_B, we get

$$\begin{aligned}|K(x,z)| &= \left|\int_0^1 1_A(z,s)\phi_B(s^{-1}(z-x)+x)s^{-n-1}ds\right| \\ &\le \|\phi_B\|_\infty \int_t^1 s^{-n-1}ds \\ &\le C(n)\rho^{-n}t^{-n} \\ &= C(n)\rho^{-n}|x-z|^{-n}(|x-x_0|+\rho)^n \\ &= C(n)\left(1+\frac{1}{\rho}|x-x_0|\right)^n |x-z|^{-n} \\ &\le C(n)\,(1+\gamma)^n\,|x-z|^{-n}.\end{aligned}$$

□

We now provide the following commutativity of Taylor polynomials and differentiation

LEMMA 1.21. *Let $A(x) = Mx + b$, be a nonsingular affine map, and let $g \in C^r(\Omega)$. Then, for any $x \in \Omega$, and $\alpha \in \mathbb{Z}_+^n$, $1 \le |\alpha| \le r$, we have*

$$D_x^\alpha\left[T_y^r(g(A\cdot))(A^{-1}x)\right] = T_y^{r-|\alpha|}(D^\alpha g(A\cdot))(A^{-1}x), \tag{1.27}$$

which implies for a star-shaped domain (with respect to B)

$$D_x^\alpha\left[T_B^r(g(A\cdot))(A^{-1}x)\right] = T_B^{r-|\alpha|}((D^\alpha g)(A\cdot)(A^{-1}x). \tag{1.28}$$

PROOF OF THEOREM 1.19. We first assume that $g \in C^r(\Omega)$ and $\operatorname{diam}(\Omega) = 1$. We require the following Riesz potential inequality: for a given $h(x) = \int_\Omega |x-z|^{r-n}|f(z)|dz$, where $f \in L_p(\Omega)$, $1 \le p \le \infty$,

$$\|h\|_{L_p(\Omega)} \le C(n,r)\operatorname{diam}(\Omega)^r\|f\|_{L_p(\Omega)}. \tag{1.29}$$

For $k = 0$, we use (1.25), (1.26) and (1.29) with $\operatorname{diam}(\Omega) = 1$, to proceed with

$$\begin{aligned}\|g - T_B^r g\|_{L_p(\Omega)} &= \|R_B^r g\|_{L_p(\Omega)} \\ &\le r\sum_{|\alpha|=r}\left\|\int_\Omega |K_\alpha(x,z)||D^\alpha g(z)|dz\right\|_{L_p(\Omega)} \\ &\le C(n,r)(\gamma+1)^n\sum_{|\alpha|=r}\left\|\int_\Omega |x-z|^{r-n}|D^\alpha g(z)|dz\right\|_{L_p(\Omega)} \\ &\le C(n,r)(\gamma+1)^n|g|_{W_p^r(\Omega)}.\end{aligned}$$

For $0 < k < r$, let $\alpha \in \mathbb{Z}_+^n$, with $|\alpha| = k$ and let $h := D^\alpha g$. Applying (1.28) with $A(x) = x$, and the case $k = 0$, for h, gives

$$\begin{aligned}\|D^\alpha(g - T_B^r g)\|_{L_p(\Omega)} &= \|h - T_B^{r-k}h\|_{L_p(\Omega)} \\ &\le C(n,r)(\gamma+1)^n|h|_{W_p^{r-k}(\Omega)} \\ &\le C(n,r)(\gamma+1)^n|g|_{W_p^r(\Omega)}.\end{aligned}$$

Summing up over all $\alpha \in \mathbb{Z}_+^n$, with $|\alpha| = k$, we conclude

$$|g - T_B^r g|_{W_p^k(\Omega)} \le C(n,r)(\gamma+1)^n|g|_{W_p^r(\Omega)}, \qquad k = 0,\dots,r-1.$$

This finishes the proof for the case $g \in C^r(\Omega)$ and $\operatorname{diam}(\Omega) = 1$. For an arbitrary bounded domain Ω that is star-shaped with respect to B, let $\tilde{\Omega} = A^{-1}(\Omega)$, where A is an affine transform defined through its inverse $A^{-1}(x) := 2\operatorname{diam}(\Omega)^{-1}(x - \tilde{x})$, where $\tilde{x} \in \Omega$ is the mid-point of the longest segment contained in Ω with length $\operatorname{diam}(\Omega)$. This gives that $\operatorname{diam}(\tilde{\Omega}) = 1$ with $\tilde{\Omega}$ star-shaped with respect to $A^{-1}(B)$, having the same chunkiness parameter γ as Ω. For $\tilde{g} := g(A\cdot)$, by the previous part in the proof

$$|\tilde{g} - T^r_{A^{-1}(B)}\tilde{g}|_{W^k_p(\tilde{\Omega})} \le C(n,r)(\gamma+1)^n |\tilde{g}|_{W^r_p(\tilde{\Omega})}, \qquad k = 0, ..., r-1.$$

Thus, with $P := T^r_{A^{-1}(B)}\tilde{g}(A^{-1}\cdot) \in \Pi_{r-1}$, we obtain for $1 \le p < \infty$ (the proof for $p = \infty$ is exactly the same with no need for the change of variables)

$$\begin{aligned}
\|g - P\|_{L_p(\Omega)} &= \left(\frac{2}{\operatorname{diam}(\Omega)}\right)^{1/p} \|\tilde{g} - T^r_{A^{-1}(B)}\tilde{g}\|_{L_p(\tilde{\Omega})} \\
&\le C(n,r)\left(\frac{2}{\operatorname{diam}(\Omega)}\right)^{1/p} (\gamma+1)^n |\tilde{g}|_{W^r_p(\tilde{\Omega})} \\
&\le C(n,r)\left(\frac{2}{\operatorname{diam}(\Omega)}\right)^{1/p} (\gamma+1)^n \operatorname{diam}(\Omega)^r \sum_{|\alpha|=r} \|D^\alpha g(A\cdot)\|_{L_p(\tilde{\Omega})} \\
&= C(n,r)(\gamma+1)^n \operatorname{diam}(\Omega)^r \sum_{|\alpha|=r} \|D^\alpha g\|_{L_p(\Omega)} \\
&= C(n,r)(\gamma+1)^n \operatorname{diam}(\Omega)^r |g|_{W^r_p(\Omega)}.
\end{aligned}$$

For $0 < k < r$, let $\alpha \in \mathbb{Z}^n_+$, with $|\alpha| = k$ and let $h := D^\alpha g$, $\tilde{h} := h(A\cdot)$. Applying (1.28) with the affine transform A defined above, and the case $k = 0$, for h, gives

$$\begin{aligned}
\|D^\alpha(g - P)\|_{L_p(\Omega)} &= \|h - D^\alpha[T^r_{A^{-1}(B)}\tilde{g}(A^{-1}\cdot)]\|_{L_p(\Omega)} \\
&= \|h - T^{r-k}_{A^{-1}(B)}\tilde{h}(A^{-1}\cdot)\|_{L_p(\Omega)} \\
&\le C(n,r)(\gamma+1)^n \operatorname{diam}(\Omega)^{r-k} |h|_{W^{r-k}_p(\Omega)} \\
&\le C(n,r)(\gamma+1)^n \operatorname{diam}(\Omega)^{r-k} |g|_{W^r_p(\Omega)}.
\end{aligned}$$

Summing up over all $\alpha \in \mathbb{Z}^n_+$, with $|\alpha| = k$, we conclude

$$|g - P|_{W^k_p(\Omega)} \le C(n,r)(\gamma+1)^n \operatorname{diam}(\Omega)^{r-k} |g|_{W^r_p(\Omega)}, \qquad k = 0, ..., r-1.$$

This concludes the proof for $g \in C^r(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W^r_p(\Omega)$, $1 \le p < \infty$, we may apply a standard density argument to obtain (1.21) for $g \in W^r_p(\Omega)$. That is, there exists sequences $\{g_k\}$, $g_k \in C^r(\Omega)$, $\{P_k\}$, $P_k \in \Pi_{r-1}$, $k \ge 1$, for which (1.21) is satisfied and also $\|g - g_k\|_{W^r_p(\Omega)} \to 0$. We may then extract a converging subsequence from $\{P_k\}$ to $P \in \Pi_{r-1}$ (e.g in the L_∞ norm), such that (1.21) is satisfied for g with P. □

The Bramble-Hilbert lemma for star-shaped domains implies that for Ω, a star-shaped domain with respect to some ball B, with chunkiness parameter γ and $f \in L_p(\Omega)$, $1 \le p \le \infty$ we have

$$K_r(f,\Omega)_p \le E_{r-1}(f,\Omega)_p \le C(n,r)(\gamma+1)^n K_r(f,\Omega)_p. \tag{1.30}$$

By (1.15) if we further assume that the domain satisfies the uniform cone condition, then the equivalence

$$E_{r-1}(f,\Omega)_p \sim K_r(f,\Omega)_p \sim \omega_r(f,\Omega)_p, \tag{1.31}$$

holds for $1 \le p \le \infty$ with constants that also depend on the shape of the domain Ω. An application of Theorem 1.19 is

THEOREM 1.22. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain and let* A *be a nonsingular affine map such that* $B(0,1) \subseteq A^{-1}(\Omega) \subseteq B(0,n)$ *and* $A^{-1}(\Omega)$ *is star-shaped with respect to* $B(0,1)$. *Then, for* $g \in W_p^r(\Omega)$, $r \ge 1$, $1 \le p \le \infty$, *there exists a polynomial* $P \in \Pi_{r-1}$, *such that*

$$|g-P|_{W_p^k(\Omega)} \le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{W_p^k(\Omega)}, \quad k=0,1,...,r. \tag{1.32}$$

For the case of $g \in C^r(\Omega)$, $P(x) = T^r_{B(0,1)}(g(A\cdot))(A^{-1}x)$, *satisfies (1.32).*

PROOF. Note that we can bound the chunkiness parameter (1.20) as follows

$$\gamma(\Omega) = \gamma(A^{-1}(\Omega)) \le 2n. \tag{1.33}$$

Since $A(x) = Mx + b$, maps $B(0,1)$ into Ω, we get that $\|M\|_2 \le \operatorname{diam}(\Omega)$. This gives that $\max_{1\le i,j\le n}|a_{i,j}| \le \operatorname{diam}\Omega$, where $M = (a_{i,j})_{1\le i,j\le n}$. With $\tilde{g} := g(A\cdot)$, and $\tilde{\Omega} := A^{-1}(\Omega)$, we get for $y \in \tilde{\Omega}$, and $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = k$, $k = 1,...,r$,

$$|D^\alpha \tilde{g}(y)| \le \operatorname{diam}(\Omega)^k \sum_{|\beta|=k} |(D^\beta g)(Ay)|.$$

In particular

$$\sum_{|\alpha|=r} \|D^\alpha \tilde{g}\|_{L_p(\tilde{\Omega})} \le c(n,r)\operatorname{diam}(\Omega)^r \sum_{|\alpha|=r} \|(D^\alpha g)(A\cdot)\|_{L_p(\tilde{\Omega})}. \tag{1.34}$$

We can now prove (1.32) for k=0. Let $\tilde{P} := T^r_{B(0,1)}\tilde{g} \in \Pi_{r-1}$ and $P := \tilde{P}(A^{-1}\cdot)$. Then, since the chunkiness parameter of $\tilde{\Omega}$ satisfies (1.33), we obtain using (1.21) and (1.34) for $1 \le p < \infty$ (the proof for $p = \infty$ is exactly the same with no need for the change of variables)

$$\begin{aligned}
\|g-P\|_{L_p(\Omega)} &= |\det M|^{1/p}\|\tilde{g}-\tilde{P}\|_{L_p(\tilde{\Omega})} \\
&\le c(n,r)|\det M|^{1/p}|\tilde{g}|_{W_p^r(\tilde{\Omega})} \\
&\le c(n,r)|\det M|^{1/p}\operatorname{diam}(\Omega)^r \sum_{|\alpha|=r}\|D^\alpha g(A\cdot)\|_{L_p(\tilde{\Omega})} \\
&= c(n,r)\operatorname{diam}(\Omega)^r \sum_{|\alpha|=r}\|D^\alpha g\|_{L_p(\Omega)} \\
&= c(n,r)\operatorname{diam}(\Omega)^r|g|_{W_p^r(\Omega)}.
\end{aligned}$$

For $0 < k < r$, we proceed as in the proof of Theorem 1.21. Let $\alpha \in \mathbb{Z}_+^n$, with $|\alpha| = k$ and let $h := D^\alpha g$, $\tilde{h} := h(A\cdot)$. Applying (1.28) with the affine transform A

defined above, and the case $k = 0$, for h, gives

$$\begin{aligned}\|D^\alpha(g-P)\|_{L_p(\Omega)} &= \|h - D^\alpha[T^r_{B(0,1)}\tilde{g}(A^{-1}\cdot)]\|_{L_p(\Omega)} \\ &= \|h - T^{r-k}_{B(0,1)}\tilde{h}(A^{-1}\cdot)\|_{L_p(\Omega)} \\ &\le C(n,r)\operatorname{diam}(\Omega)^{r-k}|h|_{W_p^{r-k}(\Omega)} \\ &\le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{W_p^r(\Omega)}.\end{aligned}$$

Summing up over all $\alpha \in \mathbb{Z}_+^n$, with $|\alpha| = k$, we conclude

$$|g-P|_{W_p^k(\Omega)} \le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{W_p^r(\Omega)}, \qquad k = 0, ..., r-1.$$

This concludes the proof for $g \in C^r(\Omega)$. Since $C^\infty(\Omega)$ is dense in $W_p^r(\Omega)$, $1 \le p < \infty$, we may apply a standard density argument to obtain (1.32) for $g \in W_p^r(\Omega)$.

□

An immediate application of John's Theorem 1.3 and Theorem 1.22 gives

Corollary 1.23. **Bramble-Hilbert Lemma for convex domains** Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $g \in W_p^r(\Omega)$, $r \in \mathbb{N}$, $1 \le p \le \infty$. Then there exists a polynomial $P \in \Pi_{r-1}$ for which

$$|g-P|_{k,p} \le C(n,r)\operatorname{diam}(\Omega)^{r-k}|g|_{r,p}, \quad k = 0, 1 \ldots, r-1. \tag{1.35}$$

For the case of $g \in C^r(\Omega)$, $P(x) = T^r_{B(0,1)}(g(A\cdot))(A^{-1}x)$, satisfies (1.35), where $T^r_B h$ is the averaged Taylor polynomial of h, with respect to a ball B, given by (1.24). In particular, for the case $k = 0$, we obtain

$$E_{r-1}(g,\Omega)_p \le C(n,r)\operatorname{diam}(\Omega)^r|g|_{r,p}. \tag{1.36}$$

We also get the following for the general case of functions in $L_p(\Omega)$,

Corollary 1.24. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $f \in L_p(\Omega)$, $1 \le p \le \infty$. Then, for any $r \ge 1$,

$$E_{r-1}(f,\Omega)_p \sim K_r(f,\Omega)_p, \tag{1.37}$$

where the constants of equivalency depend only on n and r and not on f or Ω.

Proof. Let $g_i \in W_p^r(\Omega)$, $i \ge 1$, be a sequence such that

$$K_r(f,\operatorname{diam}(\Omega)^r)_p = \inf_i\{\|f-g_i\|_p + \operatorname{diam}(\Omega)^r|g_i|_{r,p}\}.$$

By (1.35) there exist polynomials $P_i \in \Pi_{r-1}$, $i \ge 1$, such that

$$\|g_i - P_i\|_p \le C(n,r)\operatorname{diam}(\Omega)^r|g_i|_{r,p}.$$

Therefore

$$\begin{aligned}E_{r-1}(f,\Omega)_p &\le \inf_i \|f-P_i\|_p \\ &\le \inf_i\{\|f-g_i\|_p + \|g_i-P_i\|_p\} \\ &\le \inf_i\{\|f-g_i\|_p + C(n,r)\operatorname{diam}(\Omega)^r|g_i|_{r,p}\} \\ &\le C(n,r)K_r(f,\operatorname{diam}(\Omega)^r)_p \\ &= C(n,r)K_r(f,\Omega)_p.\end{aligned}$$

To prove $K_r(f,\Omega)_p \le E_{r-1}(f,\Omega)_p$, let P be an arbitrary polynomial in Π_{r-1}. Then, it is easy to see using (1.7)

$$K_r(f,\operatorname{diam}(\Omega)^r)_p \le \|f-P\|_p + \operatorname{diam}(\Omega)^r |P|_{r,p} = \|f-P\|_p.$$

Since P was chosen arbitrarily, we get that

$$K_r(f,\Omega)_p = K_r(f,\operatorname{diam}(\Omega)^r)_p \le \inf_{P\in\Pi_{r-1}} \|f-P\|_p = E_{r-1}(f,\Omega)_p.$$

□

1.5. The Whitney Theorem for convex domains

In the previous section, when the polynomial approximation was taking place in the L_p space, with $1 \le p \le \infty$, we were able to apply the tools of Sobolev spaces and the K-functional. However, for the case of $0 < p < 1$, one needs to directly estimate 'local' low order polynomial approximation over convex domains using the modulus of smoothness explicitly. The critical emphasis is on estimates where the leading constant does no depend on the geometry of the domain. The main result of this section is

THEOREM 1.25. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and let $f \in L_p(\Omega)$, $0 < p \le \infty$. Then for any $r \ge 1$*

$$E_{r-1}(f,\Omega)_p \le C(n,r,p)\omega_r(f,\Omega)_p, \tag{1.38}$$

where $\omega_r(f,\Omega)_p$ is defined in (1.11).

By the first part of Theorem 1.18 we have that $\omega_r(P,\Omega)_p = 0$, for any polynomial $P \in \Pi_{r-1}$. Thus,

$$\omega_r(f,\Omega)_p \le \omega_r(f-P,\Omega)_p \le C\|f-P\|_p,$$

which gives

$$\omega_r(f,\Omega)_p \le CE_{r-1}(f,\Omega)_p.$$

Combining this with (1.37) and (1.38) yields

COROLLARY 1.26. For all bounded convex domains $\Omega \subset \mathbb{R}^n$, and functions $f \in L_p(\Omega)$, if $1 \le p \le \infty$, then we have the equivalence

$$E_{r-1}(f,\Omega)_p \sim K_r(f,\Omega)_p \sim \omega_r(f,\Omega)_p, \tag{1.39}$$

and for $0 < p < 1$, the equivalence

$$E_{r-1}(f,\Omega)_p \sim \omega_r(f,\Omega)_p. \tag{1.40}$$

where the constants of equivalency depend only on n,r and p.

We prove Theorem 1.25 separately for $1 \le p \le \infty$, and then for $0 < p < 1$. As we shall see, in the former case, we can use the equivalence of the modulus of smoothness and the K-functional and then apply the machinery of K-functionals. In the latter case we have to work significantly harder as the classical K-functional in L_p, $0 < p < 1$, is trivial.

Proof of Theorem 1.25 for the case $1 \le p \le \infty$ Let $A(x) = Mx + b$ be the affine transform for which (1.3) holds. Corollary 1.24 implies that for $\widetilde{\Omega} := A^{-1}(\Omega)$ and $\widetilde{f} := f(A\cdot)$ there exists a polynomial $\widetilde{P} \in \Pi_{r-1}$ such that

$$\left\|\widetilde{f} - \widetilde{P}\right\|_{L_p(\widetilde{\Omega})} \le C(n,r)K_r(\widetilde{f},\widetilde{\Omega})_p.$$

Since $B(0,1) \subseteq \widetilde{\Omega} \subseteq B(0,n)$, $\widetilde{\Omega}$ fulfills the conditions of Corollary 1.13 with $R_1 = 1, R_2 = n$, we may apply (1.16) with $t = \operatorname{diam}(\widetilde{\Omega})$, to obtain

$$\begin{aligned}\left\|\tilde{f} - \tilde{P}\right\|_{L_p(\tilde{\Omega})} &\le C(n,r)K_r(\tilde{f},\tilde{\Omega})_p \\ &\le C(n,r,p)\omega_r(\tilde{f},\tilde{\Omega})_p.\end{aligned}$$

Denoting $P := \tilde{P}(A^{-1}\cdot)$, yields

$$\begin{aligned}\left\|f - P\right\|_{L_p(\Omega)} &= |\det M|^{1/p}\left\|\tilde{f} - \tilde{P}\right\|_{L_p(\tilde{\Omega})} \\ &\le C|\det M|^{1/p}\omega_r(\tilde{f},\tilde{\Omega})_p \\ &= C\omega_r(f,\Omega)_p.\end{aligned}$$

This proves Theorem 1.25 for the case $1 \le p \le \infty$. We now turn to the proof of the Whitney theorem for the case $0 < p < 1$ [**14**]. We first prove the case $r = 1$

LEMMA 1.27. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in L_p(\Omega)$, $0 < p < \infty$. Then there exists a constant c such that*

$$\int_\Omega |f(x) - c|^p dx \le \frac{1}{|\Omega|}\int_{|h|\le \operatorname{diam}(\Omega)}\int_\Omega |\Delta_h(f,\Omega,x)|^p dx dh, \tag{1.41}$$

where $|\Omega|$ denotes the volume of the domain Ω.

PROOF. By a standard density argument, one may assume that f is continuous. Consider the function $\phi(y) := \int_\Omega |f(x) - f(y)|^p\, dx$, $y \in \Omega$. Clearly, there exists $y_0 \in \Omega$ such that

$$\phi(y_0) \le \frac{1}{|\Omega|}\int_\Omega \phi(y)\, dy.$$

Therefore with $c := f(y_0)$ we get

$$\begin{aligned}\int_\Omega |f(x) - c|^p\, dx &= \phi(y_0) \\ &\le \frac{1}{|\Omega|}\int_\Omega \phi(y)\, dy \\ &= \frac{1}{|\Omega|}\int_\Omega\int_\Omega |f(x) - f(y)|^p\, dx\, dy.\end{aligned}$$

By definition, for any domain Ω and every $x \in \Omega$, if $x+h \notin \Omega$, then $\Delta_h(f,\Omega,x) = 0$. Therefore, the substitution $h = y - x$ yields (1.41). □

COROLLARY 1.28. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and $f \in L_p(\Omega)$, $0 < p < \infty$. Then there exists a constant c such that

$$\left\|f - c\right\|_{L_p(\Omega)} \le (2n)^{n/p}\omega_1(f,\Omega)_p. \tag{1.42}$$

PROOF. Let $\widetilde{\Omega} := A^{-1}(\Omega)$, where A is the affine transform for which (1.3) holds. Denote $\tilde{f} := f(A\cdot)$. By Lemma 1.27 there exists a constant c such that

$$\int_{\widetilde{\Omega}} \left|\tilde{f}(x) - c\right|^p dx \le \frac{1}{|\widetilde{\Omega}|}\int_{|h|\le 2n}\int_{\widetilde{\Omega}} |\Delta_h(\tilde{f},\widetilde{\Omega},x)|^p\, dx\, dh.$$

Hence

$$\begin{aligned}\int_{\widetilde{\Omega}} |\tilde{f}(x) - c|^p \, dx &\leq \frac{|B(0, 2n)|}{|B(0, 1)|} \omega_1(\tilde{f}, \widetilde{\Omega})_p^p \\ &= (2n)^n \omega_1(\tilde{f}, \widetilde{\Omega})_p^p.\end{aligned}$$

As we have seen in the proof of Theorem 1.25 for the case $1 \leq p \leq \infty$, the Whitney inequality is invariant under affine maps and therefore the above inequality implies (1.42). $\square$

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