

Distributions and Function Spaces

5.1. Test Functions and Distributions

DEFINITION 5.6. For a space (M, ρ, μ) , we shall denote by \mathcal{D} the **Test Functions** $\phi \in L^2$ such that $\phi \in \bigcap_{m \geq 1} D(L^m)$ with topology induced by

$$\mathcal{P}_{m,l}(\phi) := \sup_{x \in M} (1 + \rho(x, x_0))^l |L^m \phi(x)| < \infty, \quad \forall m, l \geq 0.$$

Here, $x_0 \in M$ is selected arbitrarily and fixed once and for all. As in the case of space of finite measure, it is convenient to have a directed family of norm

$$\mathcal{P}_{m,l}^*(\phi) := \max_{0 \leq r \leq m, 0 \leq s \leq l} \mathcal{P}_{r,s}(\phi).$$

Assignment - Prove that $\mathcal{D} \subset L^p$, for any $0 < p \leq \infty$.

Observe that the set \mathcal{D} and the topology induced by $\{\mathcal{P}_{m,l}^*\}$, remain unchanged if we choose a different base point $x_1 \in M$. Indeed, for any $r \leq m$, $s \leq l$ and $x \in M$

$$\begin{aligned}
(1 + \rho(x, x_1))^s |L^r \phi(x)| &\leq (1 + \rho(x, x_0) + \rho(x_0, x_1))^s |L^r \phi(x)| \\
&\leq \sum_{j=0}^s \binom{s}{j} \rho(x_0, x_1)^j (1 + \rho(x, x_0))^{s-j} |L^r \phi(x)| \\
&\leq \sum_{j=0}^s \binom{s}{j} \rho(x_0, x_1)^j \mathcal{P}_{m,l}^*(\phi) \\
&\leq c(l, \rho(x_0, x_1)) \mathcal{P}_{m,l}^*(\phi).
\end{aligned}$$

EXAMPLE 5.7. let $\varphi \in \mathcal{S}(\mathbb{R})$ be even. Then $\varphi(\sqrt{L})(x, \cdot), \varphi(\sqrt{L})(\cdot, y) \in \mathcal{D}$, for any $x, y \in M$,

PROOF. We prove that $\varphi(\sqrt{L})(x, \cdot) \in \mathcal{D}$, for any $x \in M$. The other case is similar. Let $m, l \geq 0$ and choose $l' = l + d/2$. Application of (2.38) gives that $L^m \varphi(\sqrt{L})$ is a kernel operator with a kernel satisfying

$$|L^m \varphi(\sqrt{L})(x, y)| \leq c(m, l') D_{1, l'}(x, y) \quad \forall x, y \in M.$$

Next, (1.30) along with the triangle inequality imply that with fixed $x \in M$, for any $y \in M$

$$\begin{aligned} |L^m \varphi(\sqrt{L})(x, y)| &\leq c'(m, l) |B(x, 1)|^{-1} (1 + \rho(x, y))^{-l} \\ &\leq c'(m, l) |B(x, 1)|^{-1} (1 + \rho(x_0, x))^l (1 + \rho(x_0, y))^{-l}. \end{aligned}$$

Recall that by (1.54) for any fixed $x \in M$ and $m \geq 1$

$$L^m[\varphi(\sqrt{L})(x, \cdot)] = [L^m\varphi(\sqrt{L})](x, \cdot).$$

We conclude that for any fixed $x \in M$ and $m \geq 1$

$$|L^m[\varphi(\sqrt{L})(x, \cdot)](y)| \leq c(m, l, x, x_0)(1 + \rho(x_0, y))^{-l}, \quad \forall y \in M,$$

which implies $\varphi(\sqrt{L})(x, \cdot) \in \mathcal{D}$. □

It is not so easy to see that also in the general case, the class \mathcal{D} is composed of ‘infinitely and continuously ‘differentiable’ functions

Theorem

Suppose $\sigma > 0$ and let $m \geq \sigma + d$ and $l > (d + \alpha)/2$. Then there exists a constant $c > 0$ such that for any $\phi \in \mathcal{D}$

$$|\phi(x) - \phi(y)| \leq c\mathcal{P}_{m,l}(\phi)\rho(x, y)^\alpha(1 + \rho(x, x_0))^{-\sigma}, \quad \forall x, y \in M, \rho(x, y) \leq 1.$$

EXAMPLE 5.9. In the case $M = \mathbb{R}^d$, the class of test functions \mathcal{D} derived from the operator $L = -\Delta$, is exactly the Schwartz class $\mathcal{S}(\mathbb{R}^d)$.

PROOF. Recall that $\mathcal{S}(\mathbb{R}^d)$ consists of all functions $\phi \in C^\infty(\mathbb{R}^d)$, such that

$$\|\phi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \phi(x)| < \infty, \quad \forall \alpha, \beta \in \mathbb{Z}_+^d.$$

At the same time, the class \mathcal{D} is defined with the choice of basepoint $x_0 = 0$, as all functions $\phi \in C^\infty(\mathbb{R}^d)$, such that

$$\mathcal{P}_{m,l}(\phi) := \sup_{x \in \mathbb{R}^d} (1 + |x|)^l |\Delta^m \phi(x)| < \infty, \quad \forall m, l \geq 0.$$

DEFINITION 5.4. We denote the space \mathcal{D}' of **distributions**, as the space of all continuous linear functionals on \mathcal{D} . The pairing of $f \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, will be denoted by $\langle f, \phi \rangle := f(\overline{\phi})$. This is consistent with the inner product $\langle f, g \rangle = \int_M f \overline{g} d\mu$ in L^2 .

The kernel operators of Proposition 5.3(iv), are a prototypical example of kernel operators T , where $T(x, \cdot) \in \mathcal{D}$, $\forall x \in M$. In such a case, the action of T on \mathcal{D}' can be well defined using duality

$$(5.1) \quad Tf(x) := f(T(x, \cdot)), \quad \forall f \in \mathcal{D}'.$$

Another important example of operators whose action can be extended to distributions are the projection operators E_λ , $\lambda \geq 1$, for the case $|M| < \infty$ and $L^2(M)$ separable. By Theorem 3.27, the spectrum $\sigma(L)$ is discrete and consists of an increasing sequence $0 \leq \lambda_1 < \lambda_2 < \dots$. Denoting $\mathcal{H}_j := \text{Ker}(\lambda_j I - L)$, then $1 \leq \dim(\mathcal{H}_j) < \infty$, and $L^2(M) = \bigoplus_j \mathcal{H}_j$. This implies that there exists an orthonormal basis of (polynomial) eigenfunctions $\{P_{j,l}\}$, $P_{j,l} \in D$, that allows to define

$$(5.2) \quad E_\lambda f := \sum_{\lambda_j \leq \lambda} \sum_{l=1}^{\dim(\mathcal{H}_j)} \langle f, P_{j,l} \rangle P_{j,l}, \quad \forall f \in \mathcal{D}'.$$

PROPOSITION 5.10. A linear functional f belongs to \mathcal{D}' if and only if there exist $m, l \geq 1$ and a constant $c > 0$, such that

$$(5.6) \quad |f(\phi)| \leq c\mathcal{P}_{m,l}^*(\phi), \quad \forall \phi \in \mathcal{D}.$$

PROOF. If (5.6) is satisfied, then obviously f is a bounded functional and thus $f \in \mathcal{D}'$. In the other direction, assume f is a bounded linear functional on \mathcal{D} . We remind the reader of some aspects of the locally convex topology the semi-norms $\{\mathcal{P}_{m,l}\}$ induce on \mathcal{D} . This means that basic opens sets of 0 are determined by finite subsets of semi-norms

$$U_{\varepsilon, m_1, l_1, \dots, m_k, l_k} := \{\phi \in \mathcal{D} : \mathcal{P}_{m_j, l_j}(\phi) < \varepsilon, 1 \leq j \leq k\}.$$

Since f is continuous, then there exists a neighborhood U of 0 such that if $\phi \in U$, then $|f(\phi)| \leq 1$. Let

$$0 \in U_{\varepsilon, m_1, l_1, \dots, m_k, l_k} \subseteq U,$$

for some $\varepsilon > 0$ and subset of semi-norms $\{\mathcal{P}_{m_j, l_j}\}_{j=1}^k$. Denote $m = \max_{1 \leq j \leq k} m_k$, $l = \max_{1 \leq j \leq k} l_k$ and $c := 2\varepsilon^{-1}$. For any $0 \neq \phi \in \mathcal{D}$, denote

$$\phi_\varepsilon := \frac{\varepsilon}{2\mathcal{P}_{m, l}^*(\phi)} \phi \in U_{\varepsilon, m_1, l_1, \dots, m_k, l_k}.$$

Then,

$$\begin{aligned} |f(\phi)| &= 2\varepsilon^{-1} \mathcal{P}_{m, l}^*(\phi) |f(\phi_\varepsilon)| \\ &\leq c \mathcal{P}_{m, l}^*(\phi). \end{aligned}$$

□

PROPOSITION 5.11. Assume the non-collapsing condition (1.20). Let $\varphi \in \mathcal{S}(\mathbb{R})$ be even. Then, for any $f \in \mathcal{D}'$, $\delta > 0$, $\varphi(\delta\sqrt{L})f$ has at most polynomial growth, as per Definition 3.6.

PROOF. Using Example 5.7, the kernel of $\varphi(\delta\sqrt{L})$ satisfies $\varphi(\delta\sqrt{L})(x, \cdot), \varphi(\delta\sqrt{L})(\cdot, y) \in \mathcal{D}, \forall x, y \in M$. Applying Proposition 5.10, implies there exist $m, l \geq 1$ and a constant $c > 0$, such that $\forall x \in M$

$$\begin{aligned}
|\varphi(\delta\sqrt{L})f(x)| &= |f(\varphi(\delta\sqrt{L})(x, \cdot))| \\
&\leq c\mathcal{P}_{m,l}^*(\varphi(\delta\sqrt{L})(x, \cdot)) \\
&\leq c \sup_{y \in M, 0 \leq r \leq m} (1 + \rho(y, x_0))^l |L^r [\varphi(\delta\sqrt{L})(x, \cdot)](y)| \\
&= c \sup_{y \in M, 0 \leq r \leq m} (1 + \rho(y, x_0))^l |L^r \varphi(\delta\sqrt{L})(x, y)| \\
&\leq c \max(1, \delta^{-2m}) \sup_{y \in M} (1 + \rho(y, x_0))^l D_{\delta, \sigma}(x, y),
\end{aligned}$$

where we applied (2.38) for the last inequality, choosing $\sigma > l + d/2$. By the triangle inequality and (1.21)

$$\begin{aligned} |\varphi(\delta\sqrt{L})f(x)| &\leq c_1(\delta)|B(x, \delta)|^{-1} \sum_{k=0}^l \rho(x, x_0)^k \sup_{y \in M} (1 + \rho(x, y))^{l-k} (1 + \rho(x, y))^{-(\sigma-d/2)} \\ &\leq c_2(\delta) \sum_{k=0}^l \rho(x, x_0)^k \\ &\leq c_3(\delta)(1 + \rho(x, x_0))^l. \end{aligned}$$

□

(1.21)	$\inf_{x \in M} B(x, r) \geq c_0^{-1} \hat{c} r^d, \quad 0 < r < 1.$
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