Foundations of approximation theory: Assignment III

1. Assume ϕ^*, ψ^* generate a univariate orthonormal multiresolution and a wavelet system, respectively. Prove that for $n=2$ the following is an orthonormal basis of $L_2\left(\mathbb{R}^2\right)$:

$$\left\{\psi_{j,k}^e\right\}, \qquad \psi_{j,k}^e\left(x\right):=2^j\psi^e\left(2^j x-k\right), \qquad e=1,2,3, \ j\in\mathbb{Z}, \quad k\in\mathbb{Z}^2,$$

   where

$$\psi^1\left(x_1,x_2\right):=\phi^*\left(x_1\right)\psi^*\left(x_2\right), \quad \psi^2\left(x_1,x_2\right):=\psi^*\left(x_1\right)\phi^*\left(x_2\right), \quad \psi^3\left(x_1,x_2\right):=\psi^*\left(x_1\right)\psi^*\left(x_2\right).$$

2. Give an example for a sequence $\beta=\left\{\beta_k\right\}$, where $\beta\in wl_2$, $\beta\notin l_2$
3. Prove the wavelet Jackson inequality for a general $N\geq 1$.
   Hint: Same proof with slight modifications.
4. Use the method of the wavelet Jackson inequality to prove the embedding

$$B_\tau^\alpha\left(\mathbb{R}^n\right)=B_{\tau,\tau}^\alpha\left(\mathbb{R}^n\right)\subset L_2\left(\mathbb{R}^n\right),$$

   by showing $\|f\|_2\leq C\left|f\right|_{B_\tau^\alpha}$, $\dfrac{1}{\tau}=\dfrac{\alpha}{n}+\dfrac{1}{2}$.

   Hint: Same proof with tiny modification.