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On the equivalence of the modulus of smoothness and the K-functional over convex domains

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Abstract

It is well known that for any bounded Lipschitz graph domain $\Omega \subset \mathbb{R}^d$, $r \geq 1$ and $1 \leq p \leq \infty$ there exist constants $C_1(d,r), C_2(\Omega,d,r,p) > 0$ such that for any function $f \in L_p(\Omega)$ and $t > 0$

$$C_1(d,r)\,\omega_r(f,t)_p \leq K_r\left(f,t^r\right)_p \leq C_2(\Omega,d,r,p)\,\omega_r(f,t)_p,$$

where $\omega_r(f,\cdot)_p$ is the modulus of smoothness and $K_r(f,\cdot)_p$ is the K-functional, both of order r. As can be seen, the right hand side inequality depends on the geometry of the domain. One of our main results is that there exists an absolute constant $C_3(d,r,p)$ such that for any convex domain $\Omega \subset \mathbb{R}^d$ and all functions $f \in L_p(\Omega)$, $1 \leq p \leq \infty$,

$$K_r\left(f,t^r\right)_p \leq C_3(d,r,p)\,\mu(\Omega,t)^{-(r-1+1/p)}\,\omega_r(f,t)_p,$$

where

$$\mu(\Omega,t) := \min_{x\in\Omega} \frac{|B(x,t)\cap\Omega|}{|B(0,t)|}, \quad B(x,r) := \left\{y \in \mathbb{R}^d : |x-y| \leq r\right\}.$$

For bounded convex domains, the above estimate can be improved for 'large' values of t

$$K_r\left(f,t^r\right)_p \leq C_4(d,r,p)\left(\left(1-\frac{t^r}{diam(\Omega)^r}\right)\mu(\Omega,t)^{-(r-1+1/p)}+1\right)\omega_r(f,t)_p,$$
$$0 < t \leq diam(\Omega).$$

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1. Introduction

We begin by recalling classical smoothness measures over multivariate domains. Here and throughout the paper we assume that $\Omega \subset \mathbb{R}^d$ is a closed domain with a nonempty interior. Let $W_p^r(\Omega)$, $1 \le p \le \infty$, $r \in \mathbb{N}$, denote the *Sobolev spaces* [1], namely, the spaces of functions $g \in L_p(\Omega)$ which have all their distributional derivatives of order up to r, $D^\alpha g := \frac{\partial^k g}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha \in \mathbb{Z}_+^d$, $|\alpha| := \sum_{i=1}^d \alpha_i = n$, $0 \le n \le r$, in $L_p(\Omega)$. The semi-norm of $W_p^r(\Omega)$ is given by $|g|_{r,p} := \sum_{|\alpha|=r} \|D^\alpha g\|_{L_p(\Omega)} < \infty$ and may be regarded as a measure of smoothness of order r of a function, provided it is in the appropriate Sobolev space. The *K-functional of order r* of $f \in L_p(\Omega)$, $1 \le p \le \infty$, (see e.g. [3,7]) is defined by

$$K_r(f,t)_p := K\left(f, t, L_p(\Omega), W_p^r(\Omega)\right) := \inf_{g \in W_p^r(\Omega)} \left\{ \|f-g\|_p + t\,|g|_{r,p} \right\}. \tag{1.1}$$

For $f \in L_p(\Omega)$, $0 < p \le \infty$, $h \in \mathbb{R}^d$ and $r \in \mathbb{N}$ we recall the rth order difference operator $\Delta_h^r(f) : \Omega \to \mathbb{R}$

$$\Delta_h^r(f,x) := \Delta_h^r(f,\Omega,x) := \begin{cases} \sum_{k=0}^{r} (-1)^{r+k} \binom{r}{k} f(x+kh) & [x, x+rh] \subset \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

where $[x,y]$ denotes the line segment connecting any two points $x, y \in \mathbb{R}^n$. The ***modulus of smoothness of order*** r ([4, Chapter 3], [6], [7, Chapter 2]) is defined by

$$\omega_r(f,t)_p := \sup_{|h| \le t} \left\| \Delta_h^r(f,\Omega,\cdot) \right\|_{L_p(\Omega)}, \quad t > 0, \tag{1.2}$$

where for $h \in \mathbb{R}^d$, $|h|$ denotes the Euclidian norm of h. Notice that for bounded domains with $diam(\Omega)$ as the diameter of Ω we have by definition

$$\omega_r(f,t)_p = \omega_r(f, diam(\Omega))_p, \quad t \ge diam(\Omega). \tag{1.3}$$

We recall that for $v \in \mathbb{R}^d$, $\gamma \in (0,\pi)$ and $\rho > 0$, the ***cone*** $\Gamma(v,\gamma,\rho)$,with axis v, $|v| = 1$, head angle γ and radius ρ is the set

$$\Gamma(v,\gamma,\rho) := \left\{ x \in \mathbb{R}^d : \langle x, v \rangle / |x| > \cos(\gamma),\ |x| < \rho \right\}.$$

An open subset of $\mathbb{R}^d$ is a ***Lipschitz graph*** domain if there exist constants $N \ge 1$, $\delta > 0$, open sets $\{U_i\}$, $i = 1, \ldots, N$ and cones $\Gamma_i(v_i, \gamma_i, \rho_i)$, $i = 1, \ldots, N$, that satisfy:

(1) The boundary of Ω is covered by the open sets $U_i^\delta := \{x \in U_i : B(x,\delta) \subset U_i\}$, $i = 1, \ldots, N$.
(2) For each $i = 1, \ldots, N$ and each $x \in \Omega \cap U_i$, the set $x + \Gamma_i$ is in Ω.

It is shown in [11] that if $\Omega \subset \mathbb{R}^d$ is a Lipschitz graph domain then for any $r \in \mathbb{N}$ there exist $C_1(d,r)$, $C_2(\Omega,d,r,p) > 0$ such that for any $f \in L_p(\Omega)$, $1 \le p \le \infty$,

$$C_1(d,r)\,\omega_r(f,t)_p \le K_r\left(f,t^r\right)_p \le C_2(\Omega,d,r,p)\,\omega_r(f,t)_p. \tag{1.4}$$

As we can see, the constant C_1 in (1.4) is absolute (for completeness we include a proof, see Lemma 3.1). However, in [11], the constant C_2 depends on the various Lipschitz graph parameters of Ω.

In this work we essentially show two variants of the right hand side inequality in (1.4) for the case of convex domains, where the inequality depends on the basic geometric parameters of the convex domain. Let *width* (Ω) denote the diameter of the largest d-dimensional Euclidian ball that is contained in Ω. Our first result is:

Theorem 1.1. *There exists a constant* $C(d,r,p)>0$ *such that for all bounded convex domains* $\Omega\subset\mathbb{R}^d$ *and functions* $f\in L_p(\Omega)$, $1\le p\le\infty$

$$K_r\left(f,t^r\right)_p\le C(d,r,p)\left(\frac{diam(\Omega)}{width(\Omega)}\right)^r\omega_r(f,t)_p,\quad 0<t\le diam(\Omega). \tag{1.5}$$

As an application of Theorem 1.1 we obtain:

Theorem 1.2. *There exists a constant* $C(d,r,p)>0$ *such that for all bounded convex domains* $\Omega\subset\mathbb{R}^d$ *and functions* $f\in L_p(\Omega)$, $1\le p\le\infty$

$$K_r\left(f,t^r\right)_p\le C(d,r,p)\left(\frac{diam(\Omega)^r-t^r}{width(\Omega)^r}+1\right)\omega_r(f,t)_p,\quad 0<t\le diam(\Omega). \tag{1.6}$$

In some sense, (1.6) *combines the previous estimate* (1.5) *with the fact that* [5]

$$K_r\left(f,diam(\Omega)^r\right)_{L_p(\Omega)}\sim\omega_r\left(f,diam(\Omega)\right)_{L_p(\Omega)}, \tag{1.7}$$

for all bounded convex domains with constants that depend on d, r *and* p *but not* Ω *or* f.

While these results are relatively easy to prove, they can be improved for certain types of convex domains such as long and thin boxes. Let

$$\mu(\Omega,t):=\min_{x\in\Omega}\frac{|B(x,t)\cap\Omega|}{|B(0,t)|}, \tag{1.8}$$

where $B(x,r):=\left\{y\in\mathbb{R}^d:|x-y|\le r\right\}$. The following result holds for general (possibly unbounded) convex domains.

Theorem 1.3. *There exists an absolute constant* $C(d,r,p)$ *such that any for any convex domain* $\Omega\subset\mathbb{R}^d$ *and all functions* $f\in L_p(\Omega)$, $1\le p\le\infty$,

$$K_r\left(f,t^r\right)_p\le C(d,r,p)\,\mu(\Omega,t)^{-(r-1+1/p)}\,\omega_r(f,t)_p. \tag{1.9}$$

Recall that a domain satisfies the Cone Condition (e.g. [1, *p. 82*]*) if there exists a finite cone* Γ *such that each* $x\in\Omega$ *is a vertex of a cone congruent (by rigid motion) to* Γ *and contained in* Ω. *It is easy to see that for such domains* $\mu(\Omega,t)\ge C(\gamma,d)$, *for any* $t\le\rho$, *where* γ *is the head angle of* Γ *and* $\rho>0$ *its height. In particular we have the following example.*

Corollary 1.4. *There exists an absolute constant* $C(d,r,p)$ *such that for any box*

$$\Omega=\left\{x=(x_1,\ldots,x_d)\in\mathbb{R}^d:a_i\le x_i\le b_i,1\le i\le d\right\}, \tag{1.10}$$

and $t\le width(\Omega)=\min_{1\le i\le d}(b_i-a_i)$, *we have*

$$K_r\left(f,t^r\right)_p\le C(d,r,p)\,\omega_r(f,t)_p.$$

We can also adjust (1.9) *so that it agrees with* (1.7).

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Theorem 1.5. *There exists a constant $C(d,r,p)>0$ such that for all bounded convex domains $\Omega\subset\mathbb{R}^d$ and functions $f\in L_p(\Omega)$, $1\le p\le\infty$*

$$K_r\left(f,t^r\right)_p \le C(d,r,p)\left(\left(1-\frac{t^r}{diam\,(\Omega)^r}\right)\mu(\Omega,t)^{-(r-1+1/p)}+1\right)\omega_r(f,t)_p\,,\quad 0<t\le diam\,(\Omega)\,. \tag{1.11}$$

2. Preliminaries

2.1. Some convex geometry and applications

Recall that an *ellipsoid* E is the image of the closed unit ball in $\mathbb{R}^d$ under a nonsingular affine map $A(x)=Mx+x_0$, $M\in M_{d\times d}(\mathbb{R})$, $x_0\in\mathbb{R}^d$. The *center* of E is $x_0=A(0)$. The following result by Fritz John [10] (see also [2]) is an important tool in this work.

Proposition 2.1 (*John's Theorem*). *Let $\Omega\subset\mathbb{R}^d$ be a bounded convex domain. Then there exists an ellipsoid $E\subseteq\Omega$ so that if x_0 is the center of E then the inclusions*

$$E\subseteq\Omega\subseteq x_0+d\,(E-x_0)\,,$$

hold. Here $x_0+d\,(E-x_0)$ is the set of points $\{x_0+d\,(x-x_0):x\in E\}$.

John's Theorem implies that for each convex domain Ω one can find a nonsingular affine map A such that

$$B(0,1)\subseteq\widetilde{\Omega}:=A^{-1}(\Omega)\subseteq B(0,d)\,. \tag{2.1}$$

It is interesting to note that John's ellipsoid is the ellipsoid $E\subseteq\Omega$ with maximal volume. In some sense this means that E 'covers' Ω sufficiently well. From this point on we may assume with no loss of generality that the center of John's ellipsoid, corresponding to a given convex domain, is $x_0=0$, since all analysis carried out in this paper is shift-invariant. We now show the straightforward relationships between John's transform M and the parameters $width\,(\Omega)$, $diam\,(\Omega)$.

Lemma 2.2. *Let $\Omega\subset\mathbb{R}^d$ be a bounded convex domain and let $M\in M_{d\times d}(\mathbb{R})$ such that $B(0,1)\subseteq M^{-1}(\Omega)\subseteq B(0,d)$. Then,*

$$\frac{diam\,(\Omega)}{2d}\le\|M\|_{l^2\to l^2}\le diam\,(\Omega)\,, \tag{2.2}$$

$$\frac{2}{width\,(\Omega)}\le\left\|M^{-1}\right\|_{l^2\to l^2}\le\frac{2d}{width\,(\Omega)}. \tag{2.3}$$

Proof. Let $x\in\mathbb{R}^d$ such that $|x|=1$. From the condition $B(0,1)\subseteq M^{-1}(\Omega)$ we get that $Mx\in\Omega$. Then, since Ω is convex we have that $|Mx|=|Mx-M0|\le diam\,(\Omega)$. This proves the right hand side of (2.2). To prove the left hand side, let $diam\,(\Omega)=|x-y|$, $x,y\in\partial\Omega$. Then, since $M^{-1}(\Omega)\subseteq B(0,d)$

$$\begin{aligned}|x-y| &= \left|M\left(M^{-1}x-M^{-1}y\right)\right| \\ &\le \|M\|_{l^2\to l^2}\left|M^{-1}x-M^{-1}y\right| \\ &\le 2d\,\|M\|_{l^2\to l^2}\,.\end{aligned}$$

To prove the right hand side of (2.3), let $x_1, x_2 \in \partial B_{\max}(\Omega)$ be the two intersection points of an arbitrary line going through the center of $B_{\max}(\Omega)$, the maximal Euclidian ball contained in Ω. Then $|x_1 - x_2| = width(\Omega)$. From the condition $M^{-1}(\Omega) \subseteq B(0,d)$ we have that $M^{-1}x_1, M^{-1}x_2 \in B(0,d)$ and thus $\left|M^{-1}(x_1 - x_2)\right| \le 2d$. Evidently, since this holds for all such pairs of points, this gives the right hand side of (2.3). Finally, in the other direction, observe that $M^{-1}B\left(0, \left\|M^{-1}\right\|_{l_2\to l_2}^{-1}\right) \subseteq B(0,1) \subseteq \tilde{\Omega}$ which implies that $B\left(0, \left\|M^{-1}\right\|_{l_2\to l_2}^{-1}\right) \subseteq \Omega$. But this means that $width(\Omega) \ge 2\left\|M^{-1}\right\|_{l_2\to l_2}^{-1}$ which gives the left hand side of (2.3). $\diamond$

Remark 2.3. There is another equivalent way to define the width of a convex body as the minimal distance between 2 parallel supporting hyper-planes of Ω (see e.g. [12]). Denoting this quantity $width_2(\Omega)$, it is evident that $width(\Omega) \le width_2(\Omega)$. In the other direction, using John's theorem, $width_2(\Omega)$ is certainly smaller than $2d\lambda_{\min}$ where $\lambda_{\min}$ is the smallest half-axis of John's ellipsoid corresponding to Ω. It is easy to prove that $\lambda_{\min}$ is equivalent to $\left\|M^{-1}\right\|_{l_2\to l_2}^{-1}$ and so by (2.3), $width_2(\Omega) \le C(d)\, width(\Omega)$.

Although the equivalence of the modulus of smoothness and the K-functional (1.4) depends in general on the shape of the domain, it is shown in [5] that one can provide uniform equivalency constants for a class of domains that are of roughly the same 'geometry'. A special case of [5, Lemma 2.4] yields (see also Remark 2.5).

Proposition 2.4. *There exists a constant $C(r,d,p) > 0$ such that for any convex domain $\tilde{\Omega} \subset \mathbb{R}^d$, $B(0,1) \subseteq \tilde{\Omega} \subseteq B(0,d)$, and $t > 0$*

$$K_r\left(f, t^r\right)_{L_p\left(\tilde{\Omega}\right)} \le C(d,r,p)\,\omega_r(f,t)_{L_p\left(\tilde{\Omega}\right)}. \tag{2.4}$$

2.2. The endpoint $t = diam(\Omega)$ for bounded convex domains

Let $\Pi_{r-1} := \Pi_{r-1}\left(\mathbb{R}^d\right)$ denote the multivariate polynomials of total degree $r-1$ (order r) in d variables. Given $\Omega \subset \mathbb{R}^d$, define the degree of approximation of a function $f \in L_p(\Omega)$, $0 < p \le \infty$, by

$$E_{r-1}(f,\Omega)_p := \inf_{P\in\Pi_{r-1}} \|f - P\|_{L_p(\Omega)}.$$

In [5] it is proved that for all bounded convex domains $\Omega \subset \mathbb{R}^d$ and functions $f \in L_p(\Omega)$, $1 \le p \le \infty$, we have the equivalence

$$E_{r-1}(f,\Omega)_p \sim K_r\left(f, diam(\Omega)^r\right)_p \sim \omega_r(f, diam(\Omega))_p, \tag{2.5}$$

where the constants of equivalency depend only on d, r and p. We see that at the parameter $t = diam(\Omega)$, the equivalence does not depend on the 'chunkiness' of the convex domain. Note that for $0 < p < 1$ we only have [5]

$$E_{r-1}(f,\Omega)_p \sim \omega_r(f, diam(\Omega))_p, \tag{2.6}$$

since the K-functional is unsuitable as a measure of smoothness if $0 < p < 1$ [8].

Remark 2.5. Originally (2.4) was proved for $0 < t \le diam\left(\tilde{\Omega}\right)$, but we require the result for any t. This is easy, since for any bounded convex domain $\Omega \subset \mathbb{R}^d$, $f \in L_p(\Omega)$ and

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$t \geq diam(\Omega)$, we can choose by (2.5) $P \in \Pi_{r-1}$ such that $\|f-P\|_p \leq C(d,r,p)\,\omega_r(f,\Omega)_p$. Thus,

$$K_r\left(f,t^r\right)_p \leq \|f-P\|_p \leq C(d,r,p)\,\omega_r(f,diam(\Omega))_p \leq C(d,r,p)\,\omega_r(f,t)_p.$$

3. Proofs of the main results

The left hand side of (1.4) for general domains is well known but we show it for the sake of completeness.

Lemma 3.1. *For any domain $\Omega \subset \mathbb{R}^d$ and $f \in L_p(\Omega)$, $1 \leq p \leq \infty$*

$$\omega_r(f,t)_p \leq C(r,d)\,K_r\left(f,t^r\right)_p. \tag{3.1}$$

Proof. For any $\varepsilon > 0$, there exists by definition $g \in W_p^r(\Omega)$ such that

$$\|f-g\|_p + t^r |g|_{r,p} \leq K_r\left(f,t^r\right)_p + \varepsilon.$$

We recall the following well known inequality for functions in the Sobolev space (see e.g. [7, Section 2.7] for the case of a univariate interval. The proof for a multivariate domain is similar.)

$$\omega_r(g,t)_{L_p(\Omega)} \leq C(d,r)\,t^r |g|_{W_p^r(\Omega)}. \tag{3.2}$$

Applying well known properties of the modulus of smoothness and (3.2) yields

$$\begin{aligned}\omega_r(f,t)_p &\leq \omega_r(f-g,t)_p + \omega_r(g,t)_p \\ &\leq 2^r \|f-g\|_p + Ct^r |g|_{p,r} \\ &\leq C\left(K\left(f,t^r\right)_p + \varepsilon\right).\end{aligned}$$

By letting $\varepsilon \to 0$ we obtain (3.1). ◇

Proof of Theorem 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain and let $f \in L_p(\Omega)$, $1 \leq p \leq \infty$. As noted above, without loss of generality we may assume that the 'center' of John's ellipsoid E is at zero such that the affine transform that satisfies $A(B(0,1)) = E$ is given by $Ax = Mx$ with $M \in M_{d\times d}(\mathbb{R})$. Denote $\tilde{\Omega} := A^{-1}(\Omega)$ and $\tilde{f}(x) := f(Mx)$. For $t > 0$ and an arbitrary $\varepsilon > 0$, let $\tilde{g} \in W_p^r\left(\tilde{\Omega}\right)$ such that

$$\left\|\tilde{f}-\tilde{g}\right\|_{L_p\left(\tilde{\Omega}\right)} + \left(\frac{t}{width(\Omega)}\right)^r |\tilde{g}|_{W_p^r\left(\tilde{\Omega}\right)} \leq K_r\left(\tilde{f},\left(\frac{t}{width(\Omega)}\right)^r\right)_{L_p\left(\tilde{\Omega}\right)} + \frac{\varepsilon}{\det(M)^{1/p}}.$$

Observe that with $g(x) := \tilde{g}\left(M^{-1}x\right)$ for any $\alpha \in \mathbb{Z}_+^d$, $|\alpha| = r$, with $1 \leq p < \infty$, we may apply (2.3) to obtain

$$\begin{aligned}\left\|D^\alpha g\right\|_{L_p(\Omega)} &= \det(M)^{1/p}\left(\int_{\tilde{\Omega}}\left|D^\alpha\left(\tilde{g}\left(M^{-1}\cdot\right)\right)(Mx)\right|^p \mathrm{d}x\right)^{1/p} \\ &\leq \det(M)^{1/p}\left\|M^{-1}\right\|_{l^2\to l^2}^r\left(\int_{\tilde{\Omega}}\left|D^\alpha \tilde{g}(x)\right|^p \mathrm{d}x\right)^{1/p} \\ &\leq (2d)^r \det(M)^{1/p}\, width(\Omega)^{-r}\left\|D^\alpha \tilde{g}\right\|_{L_p\left(\tilde{\Omega}\right)}.\end{aligned} \tag{3.3}$$

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With the obvious modification, (3.3) also holds for $p = \infty$. We require the following property of the modulus of smoothness (see [7, Section 2.7]). For any $\lambda > 0$

$$\omega_r (f, \lambda t)_p \le (\lambda + 1)^r \omega_r (f, t)_p . \tag{3.4}$$

The estimate (3.3) and (2.4), the right hand side of (2.2) and (3.4) yield

$$\begin{aligned}
K_r \left(f, t^r\right)_{L_p(\Omega)} &\le \|f - g\|_{L_p(\Omega)} + t^r |g|_{W_p^r(\Omega)} \\
&\le C \det (M)^{1/p} \left(\left\| \tilde{f} - \tilde{g} \right\|_{L_p\left(\tilde{\Omega}\right)} + \left(\frac{t}{width(\Omega)} \right)^r |\tilde{g}|_{W_p^r\left(\tilde{\Omega}\right)} \right) \\
&\le C \det (M)^{1/p} K_r \left(\tilde{f}, \left(\frac{t}{width(\Omega)} \right)^r \right)_{L_p\left(\tilde{\Omega}\right)} + C\varepsilon \\
&\le C \det (M)^{1/p} \omega_r \left(\tilde{f}, \frac{t}{width(\Omega)} \right)_{L_p\left(\tilde{\Omega}\right)} + C\varepsilon \\
&= C \det (M)^{1/p} \sup_{|h| \le t/width(\Omega)} \left\| \Delta_h^r \left(\tilde{f}, \cdot \right) \right\|_{L_p\left(\tilde{\Omega}\right)} + C\varepsilon \\
&= C \det (M)^{1/p} \sup_{|h| \le t/width(\Omega)} \left\| \Delta_{Mh}^r (f, M \cdot) \right\|_{L_p\left(\tilde{\Omega}\right)} + C\varepsilon \\
&\le C \omega_r \left(f, \frac{diam(\Omega)}{width(\Omega)} t \right)_{L_p(\Omega)} + C\varepsilon \\
&\le C \left(\frac{diam(\Omega)}{width(\Omega)} + 1 \right)^r \omega_r (f, t)_{L_p(\Omega)} + C\varepsilon \\
&\le C \left(\frac{diam(\Omega)}{width(\Omega)} \right)^r \omega_r (f, t)_{L_p(\Omega)} + C\varepsilon. \quad \diamond
\end{aligned}$$

Proof of Theorem 1.2. Fix $0 < t \le diam(\Omega)$. For $\varepsilon > 0$, let $g \in W_p^r(\Omega)$ such that

$$\|f - g\|_p + t^r |g|_{r,p} \le K_r \left(f, t^r\right)_p + \varepsilon.$$

From (2.5) we know that there exists a polynomial $P \in \Pi_{r-1}$, such that

$$\|f - P\|_{L_p(\Omega)} \le C(d, r, p) \omega_r (f, diam(\Omega))_p . \tag{3.5}$$

Define

$$g_t := \left(1 - \left(\frac{t}{diam(\Omega)} \right)^r \right) g + \left(\frac{t}{diam(\Omega)} \right)^r P. \tag{3.6}$$

Then, applications of (1.5) and (3.5) and then (3.4) yield

$$\begin{aligned}
K_r \left(f, t^r\right)_p &\le \|f - g_t\|_p + t^r |g_t|_{r,p} \\
&\le \left(1 - \left(\frac{t}{diam(\Omega)} \right)^r \right) K_r \left(f, t^r\right)_p + \left(\frac{t}{diam(\Omega)} \right)^r \|f - P\|_p + \varepsilon \\
&\le C \Bigg(\left(1 - \left(\frac{t}{diam(\Omega)} \right)^r \right) \left(\frac{diam(\Omega)}{width(\Omega)} \right)^r \omega_r (f, t)_p \\
&\quad + \left(\frac{t}{diam(\Omega)} \right)^r \omega_r (f, diam(\Omega))_p \Bigg) + \varepsilon
\end{aligned}$$

$$\le C\left(\frac{diam\,(\Omega)^r - t^r}{width\,(\Omega)^r}\omega_r\,(f,t)_p + \left(\frac{t}{diam\,(\Omega)}\right)^r \times \left(\frac{diam\,(\Omega)}{t}+1\right)^r \omega_r\,(f,t)_p\right) + \varepsilon$$

$$\le C\left(\frac{diam\,(\Omega)^r - t^r}{width\,(\Omega)^r}+1\right)\omega_r\,(f,t)_p + \varepsilon. \quad \diamond$$

The following two results are crucial for the proof of Theorem 1.3. The first is essentially [9, Theorem 7.1], but slightly re-formulated.

Proposition 3.2. *Suppose the following conditions hold for a convex domain $\Omega \subset \mathbb{R}^d$:*

(a) *There exist convex sets $\tilde{\Omega}_k$, $k \in I$, where I is some countable index set, such that $\Omega = \bigcup_k \tilde{\Omega}_k$.*
(b) *Each point $x \in \Omega$ is in at most N sets $\tilde{\Omega}_k$.*
(c) *There exist $t > 0$, $\tilde{\mu} > 0$ and $L > 0$ independent of k such that $\left|\tilde{\Omega}_k\right| > \tilde{\mu} t^d$ and $\tilde{\Omega}_k \subseteq B\,(x_k, Lt)$ for some $x_k \in \tilde{\Omega}_k$.*

Then, for $f \in L_p\,(\Omega)$, $0 < p < \infty$

$$\sum_k \omega_r\left(f, diam\left(\tilde{\Omega}_k\right)\right)^p_{L_p\left(\tilde{\Omega}_k\right)} \le \frac{C\,(d,r,p,N,L)}{\tilde{\mu}}\omega_r\,(f,t)^p_{L_p(\Omega)}\,, \tag{3.7}$$

and for $p = \infty$

$$\sup_{k\in I}\omega_r\left(f, diam\left(\tilde{\Omega}_k\right)\right)_{L_\infty\left(\tilde{\Omega}_k\right)} \le C\,(d,r,p,L)\,\omega_r\,(f,t)_{L_\infty(\Omega)}\,. \tag{3.8}$$

We also require the Markov inequality for polynomials over convex domains [13]. *We note that there is still active research regarding the constant in the inequality (see* [12] *and references therein).*

Proposition 3.3. *Let $\tilde{\Omega} \subset \mathbb{R}^d$ be a bounded convex domain. Then for $1 \le p \le \infty$, any polynomial $P \in \Pi_{r-1}$ and any $\beta \in \mathbb{Z}^d_+$, $|\beta| \le r-1$,*

$$\left\|D^\beta P\right\|_{L_p\left(\tilde{\Omega}\right)} \le \frac{C\,(d,r)}{width\left(\tilde{\Omega}\right)^{|\beta|}}\,\|P\|_{L_p\left(\tilde{\Omega}\right)}\,. \tag{3.9}$$

Proof of Theorem 1.3. For $f \in L_p\,(\Omega)$, $t > 0$ and a fixed constant $0 < \delta < 1$ (to be determined later) we shall construct an appropriate $g \in W^r_p\,(\Omega)$ that satisfies

$$\|f-g\|_p + (\delta t)^r\,|g|_{r,p} \le C\,(r,d,p)\,\mu\,(\Omega,\delta t)^{-(r-1+1/p)}\,\omega_r\,(f,\delta t)_p\,. \tag{3.10}$$

To this end, we subdivide $\mathbb{R}^d$ by a uniform grid of cubes of length t

$$\Box_k := \left\{x \in \mathbb{R}^d : tk_i \le x_i < t\,(k_i+1)\,, 1 \le i \le d\right\}, \quad k \in \mathbb{Z}^d,$$

and denote

$$\Omega_k := \Omega \cap \Box_k, \quad k \in \mathbb{Z}^d.$$

Evidently, the domains Ω_k are either convex or empty sets and so we denote $I := \left\{k \in \mathbb{Z}^d : \Omega_k \neq \emptyset\right\}$. We then construct an 'overlapping' grid of cubes which also gives a cover of Ω by convex subdomains (both of nonempty elements)

$$\tilde{\Box}_k := \left\{x \in \mathbb{R}^d : t(k_i - 1) \le x_i < t(k_i + 2), 1 \le i \le d\right\}, \quad \tilde{\Omega}_k := \Omega \cap \tilde{\Box}_k, \ k \in I.$$

By (2.5), for each $k \in I$ there exists a polynomial $P_k \in \Pi_{r-1}$, such that

$$\|f - P_k\|_{L_p\left(\tilde{\Omega}_k\right)} \le C\omega_r\left(f, diam\left(\tilde{\Omega}_k\right)\right)_{L_p\left(\tilde{\Omega}_k\right)} \le C\omega_r(f,t)_{L_p\left(\tilde{\Omega}_k\right)}. \tag{3.11}$$

We require an appropriate 'partition of unity' of the domain Ω that is based on our uniform grid of length t. Let $\psi : [0, 1/2] \to [0, 1]$ be a C^r function with the following properties:

(a) $0 \le \psi(s) \le 1$, $\psi(0) = 0$, $\psi(1/2) = 1$.
(b) $\psi^{(j)}(0) = \psi^{(j)}(1/2) = 0$, $1 \le j \le r$.
(c) $\left|\psi^{(j)}(s)\right| \le C_j$, $1 \le j \le r$.

The following C^r univariate functions are supported in $[-1/4, 5/4]$

$$\tilde{\phi}_1(s) := \begin{cases} 0 & s < -1/4, \\ \psi(s + 1/4) & -1/4 \le s < 1/4, \\ 1 & 1/4 \le s < 3/4, \\ 1 - \psi(s - 3/4) & 3/4 \le s < 5/4, \\ 0 & s \ge 5/4, \end{cases}$$

$$\tilde{\phi}_2(s) := \begin{cases} 0 & s < -1/4, \\ 1 & -1/4 \le s < 3/4, \\ 1 - \psi(s - 3/4) & 3/4 \le s < 5/4, \\ 0 & s \ge 5/4, \end{cases}$$

$$\tilde{\phi}_3(s) := \begin{cases} 0 & s < -1/4, \\ \psi(s + 1/4) & -1/4 \le s < 1/4, \\ 1 & 1/4 \le s < 5/4, \\ 0 & s \ge 5/4, \end{cases} \qquad \tilde{\phi}_4(s) := \begin{cases} 1 & -1/4 \le s \le 5/4, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that with an appropriate choice $\left\{\tilde{\phi}_{j(k,i)}\right\}$, $j = 1, \ldots, 4$, $k \in I$, $i = 1, \ldots, d$, the functions

$$\phi_k(x) := \prod_{i=1}^{d} \tilde{\phi}_{j(k,i)}(x_i/t - k_i), \quad k \in I,$$

satisfy a smooth partition of unity over Ω

$$\sum_{k \in I} \phi_k(x) = 1, \quad x \in \Omega.$$

Indeed, for $k = (k_1, \ldots, k_d) \in I$ and $1 \le i \le d$, there are four cases:

(1) If $(k_1, \ldots, k_{i-1}, k_i - 1, k_i, \ldots, k_d) \in I$ and $(k_1, \ldots, k_{i-1}, k_i + 1, k_i, \ldots, k_d) \in I$, then $j(k,i) = 1$,
(2) If $(k_1, \ldots, k_{i-1}, k_i - 1, k_i, \ldots, k_d) \notin I$ and $(k_1, \ldots, k_{i-1}, k_i + 1, k_i, \ldots, k_d) \in I$, then $j(k,i) = 2$,

(3) If $(k_1, \ldots, k_{i-1}, k_i - 1, k_i, \ldots, k_d) \in I$ and $(k_1, \ldots, k_{i-1}, k_i + 1, k_i, \ldots, k_d) \notin I$, then $j(k, i) = 3$,
(4) If $(k_1, \ldots, k_{i-1}, k_i - 1, k_i, \ldots, k_d) \notin I$ and $(k_1, \ldots, k_{i-1}, k_i + 1, k_i, \ldots, k_d) \notin I$, then $j(k, i) = 4$.

Thus, if the cube $\square_k$ is at the 'core' of the domain then $\phi_k(x) = \prod_{i=1}^d \tilde{\phi}_1(x_i/t - k_i)$, while on cubes near the boundary we may have $\phi_k(x) := \prod_{i=1}^d \tilde{\phi}_{j(k,i)}(x_i/t - k_i)$, with some $j(k, i) \neq 1$.

We combine the local polynomial approximation with the partition of unity to construct $g \in C^r(\Omega)$

$$g(x) := \sum_{k \in I} \phi_k(x) P_k(x) \quad x \in \Omega. \tag{3.12}$$

For g defined in (3.12) we first estimate the first term on the left hand side of (3.10). For each $k \in I$, let $\Lambda_k := \left\{ j \in I : \tilde{\Omega}_j \cap \tilde{\Omega}_k \neq \emptyset \right\}$. We use the partition of unity, the bound $\#\Lambda_k \le 3^d$ and (3.11) to derive for $1 \le p < \infty$

$$\begin{aligned}
\|f - g\|_{L_p(\Omega)}^p &= \sum_{k \in I} \int_{\Omega_k} |f - g|^p \\
&= \sum_{k \in I} \int_{\Omega_k} \left| \sum_{j \in \Lambda_k} \phi_j \left(f - P_j\right) \right|^p \\
&\le C(d, p) \sum_{k \in I} \sum_{j \in \Lambda_k} \int_{\Omega_k} \left|\phi_j \left(f - P_j\right)\right|^p \\
&\le C(d, r, p) \sum_{k \in I} \omega_r(f, t)_{L_p\left(\tilde{\Omega}_k\right)}^p .
\end{aligned}$$

For $k \in I$, let $x_k \in \Omega_k$ (recall that Ω_k is not an empty set for $k \in I$). Since $x_k \in \square_k$ we have that $B(x_k, t) \subset \tilde{\square}_k$ and therefore $B(x_k, t) \cap \Omega \subset \tilde{\square}_k \cap \Omega = \tilde{\Omega}_k$. Together with (1.8) this gives

$$\begin{aligned}
\left|\tilde{\Omega}_k\right| &\ge \left|B(x_k, t) \cap \tilde{\Omega}_k\right| \\
&= |B(x_k, t) \cap \Omega| \\
&\ge \mu(\Omega, t) |B(0, t)| \\
&\ge C(d) t^d \mu(\Omega, t) .
\end{aligned} \tag{3.13}$$

Therefore with $\tilde{\mu} := C(d) \mu(\Omega, t)$ and $L = 3\sqrt{d}/2$, the convex sets $\left\{\tilde{\Omega}_k\right\}$ satisfy the conditions of Proposition 3.2 which implies by (3.4) for $1 \le p < \infty$

$$\begin{aligned}
\|f - g\|_{L_p(\Omega)}^p &\le C(d, r, p) \sum_{k \in I} \omega_r(f, t)_{L_p\left(\tilde{\Omega}_k\right)}^p \\
&\le \frac{C(d, r, p)}{\mu(\Omega, t)} \omega_r(f, t)_{L_p(\Omega)}^p \\
&\le \frac{C(d, r, p)}{\mu(\Omega, \delta t)} \omega_r(f, \delta t)_{L_p(\Omega)}^p ,
\end{aligned}$$

where we also used in the last inequality the estimate $\mu(\Omega, \delta t) \le C(d, \delta) \mu(\Omega, t) = C(d) \mu(\Omega, t)$, for fixed $0 < \delta \le 1$.

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For $p=\infty$ we have

$$\begin{aligned}\|f-g\|_{L_\infty(\Omega)} &= \max_{k\in I}\|f-g\|_{L_\infty(\Omega_k)}\\ &= \max_{k\in I}\left\|\sum_{j\in\Lambda_k}\phi_j\left(f-P_j\right)\right\|_{L_\infty(\Omega_k)}\\ &\le C\max_{k\in I}\omega_r(f,t)_{L_\infty\left(\tilde{\Omega}_k\right)}\\ &\le C\omega_r(f,\delta t)_{L_\infty(\Omega)}.\end{aligned}$$

We now estimate $(\delta t)^r|g|_{r,p}$, which is the second term on the left hand side of (3.10). Recall that on each Ω_k, $k\in I$, one has the representation $g=\sum_{j\in\Lambda_k}\phi_j P_j$ with $\sum_{j\in\Lambda_k}\phi_j=1$. It is important that this representation contains only terms with polynomials that were constructed over the subdomains $\tilde{\Omega}_j$, $j\in\Lambda_k$ which completely contain Ω_k. For each k, we enumerate $\Lambda_k=\left\{j_1,j_2,\ldots,j_{J_k}\right\}$, so that $\#\Lambda_k=J_k$.

Let $\alpha\in\mathbb{Z}_+^d$, $|\alpha|=r$. If $J_k=1$, then $\Lambda_k=\{k\}$ which implies that $\Omega_k=\Omega$. Therefore, $\phi_k(x)=\prod_{i=1}^d\tilde{\phi}_4(x_i/t-k)=1$ on Ω_k and $D^\alpha g(x)=D^\alpha P_k(x)=0$. Else, if $J_k\ge 2$ we rewrite g on Ω_k as

$$\begin{aligned}g &= \left(1-\sum_{m=2}^{J_k}\phi_{j_m}\right)P_k+\sum_{m=2}^{J_k}\phi_{j_m}P_{j_m}\\ &= P_k+\sum_{m=2}^{J_k}\phi_{j_m}\left(P_{j_m}-P_k\right).\end{aligned}$$

Thus, if $J_k\ge 2$ one has for $1\le p<\infty$

$$\left\|D^\alpha g\right\|_{L_p(\Omega_k)}^p\le C(d,p)\sum_{m=2}^{J_k}\left\|D^\alpha\left(\phi_{j_m}\left(P_{j_m}-P_k\right)\right)\right\|_{L_p(\Omega_k)}^p. \tag{3.14}$$

We estimate each term on the right hand side of (3.14) by

$$\begin{aligned}&\left\|D^\alpha\left(\phi_{j_m}\left(P_{j_m}-P_k\right)\right)\right\|_{L_p(\Omega_k)}^p\\ &\le C(d,r,p)\sum_{\beta_1+\beta_2=\alpha,|\beta_2|\le r-1}\left\|D^{\beta_1}\phi_{j_m}D^{\beta_2}\left(P_{j_m}-P_k\right)\right\|_{L_p(\Omega_k)}^p\\ &\le C(d,r,p,\psi)\sum_{\beta_1+\beta_2=\alpha,|\beta_2|\le r-1}t^{-|\beta_1|p}\left\|D^{\beta_2}\left(P_{j_m}-P_k\right)\right\|_{L_p\left(\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right)}^p.\end{aligned} \tag{3.15}$$

Assume for a moment that for a fixed $0<\delta<1$ there exists a point $x_{m,k}\in\Omega$ such that $B\left(x_{m,k},\delta t\right)\subseteq\tilde{\Box}_{j_m}\cap\tilde{\Box}_k$. This gives

$$\begin{aligned}\left|\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right| &\ge \left|B\left(x_{m,k},\delta t\right)\cap\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right|\\ &= \left|B\left(x_{m,k},\delta t\right)\cap\Omega\right|\\ &\ge C(d)t^d\mu(\Omega,\delta t).\end{aligned} \tag{3.16}$$

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Recalling that $width_2\left(\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right)$ is the minimal distance between 2 parallel supporting hyperplanes of $\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k$ (see Remark 2.3) we have by (3.16)

$$\begin{aligned} width_2\left(\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right) &= t^{-(d-1)}width_2\left(\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right)t^{d-1}\\ &\ge t^{-(d-1)}C(d)\left|\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right|\\ &\ge t^{-(d-1)}C(d)\,t^d\mu(\Omega,\delta t)\\ &= C(d)\,t\mu(\Omega,\delta t). \end{aligned} \tag{3.17}$$

For any $\beta\in\mathbb{Z}_+^d$, $|\beta|\le r-1$, application of (3.9), (3.17) and (3.11) give

$$\begin{aligned} &\left\|D^\beta\left(P_{j_m}-P_k\right)\right\|^p_{L_p\left(\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right)} \le \frac{C}{t^{|\beta|p}\mu(\Omega,\delta t)^{|\beta|p}}\left\|P_{j_m}-P_k\right\|^p_{L_p\left(\tilde{\Omega}_{j_m}\cap\tilde{\Omega}_k\right)}\\ &\quad\le \frac{C}{t^{|\beta|p}\mu(\Omega,\delta t)^{|\beta|p}}\left(\left\|f-P_{j_m}\right\|^p_{L_p\left(\tilde{\Omega}_{j_m}\right)}+\left\|f-P_k\right\|^p_{L_p\left(\tilde{\Omega}_k\right)}\right)\\ &\quad\le \frac{C}{t^{|\beta|p}\mu(\Omega,\delta t)^{|\beta|p}}\left(\omega_r(f,t)^p_{L_p\left(\tilde{\Omega}_{j_m}\right)}+\omega_r(f,t)^p_{L_p\left(\tilde{\Omega}_k\right)}\right). \end{aligned}$$

Combining this last estimate with (3.14) and (3.15) and Proposition 3.2 yields

$$\begin{aligned} (\delta t)^{rp}|g|^p_{W_p^r(\Omega)} &\le Ct^{rp}\sum_{|\alpha|=r}\sum_{k\in I,J_k\ne1}\left\|D^\alpha g\right\|^p_{L_p(\Omega_k)}\\ &\le Ct^{rp}\sum_{|\alpha|=r}\sum_{k\in I,J_k\ne1}\sum_{m=2}^{J_k}\frac{1}{t^{rp}\mu(\Omega,\delta t)^{(r-1)p}}\\ &\quad\times\left(\omega_r(f,t)^p_{L_p\left(\tilde{\Omega}_{j_m}\right)}+\omega_r(f,t)^p_{L_p\left(\tilde{\Omega}_k\right)}\right)\\ &\le \frac{C}{\mu(\Omega,\delta t)^{(r-1)p}}\sum_{k\in I}\omega_r(f,t)^p_{L_p\left(\tilde{\Omega}_k\right)}\\ &\le \frac{C}{\mu(\Omega,\delta t)^{(r-1)p+1}}\omega_r(f,\delta t)^p_{L_p(\Omega)}. \end{aligned}$$

For $p=\infty$ we compute as in (3.14)–(3.17) to obtain

$$\begin{aligned} (\delta t)^r\left\|D^\alpha g\right\|_{L_\infty(\Omega)} &\le C\max_{k\in I}t^r\left\|D^\alpha g\right\|_{L_\infty(\Omega_k)}\\ &\le Ct^r\max_{k\in I,j\in\Lambda_k}\left\|D^\alpha\left(\phi_j\left(P_j-P_k\right)\right)\right\|_{L_\infty(\Omega_k)}\\ &\le Ct^r\max_{k\in I,j\in\Lambda_k,\beta_1+\beta_2=\alpha,|\beta_2|\le r-1}t^{-|\beta_1|}\left\|D^{\beta_2}\left(P_j-P_k\right)\right\|_{L_\infty(\Omega_k)}\\ &\le C\max_{k\in I,j\in\Lambda_k}\frac{1}{\mu(\Omega,\delta t)^{r-1}}\left\|P_j-P_k\right\|_{L_\infty\left(\tilde{\Omega}_j\cap\tilde{\Omega}_k\right)}\\ &\le C\frac{1}{\mu(\Omega,\delta t)^{r-1}}\max_{k\in I,j\in\Lambda_k}\left(\omega_r(f,t)_{L_\infty\left(\tilde{\Omega}_j\right)}+\omega_r(f,t)_{L_\infty\left(\tilde{\Omega}_k\right)}\right)\\ &\le C\frac{1}{\mu(\Omega,\delta t)^{r-1}}\omega_r(f,\delta t)_{L_\infty(\Omega)}. \end{aligned}$$

This concludes the proof of (3.10). To complete the proof of Theorem 1.3, we show that for $\delta = 1/2$ and any neighboring cubes $\square_j$ and $\square_k$, $j,k \in I$, there exists a point $x_{j,k} \in \Omega$ satisfying $B\left(x_{j,k}, \delta t\right) \subseteq \tilde{\square}_j \cap \tilde{\square}_k$. (We thank the referee for providing a significantly shorter proof of this part.) Since our construction guarantees that there exist $x_j \in \Omega_j = \square_j \cap \Omega$ and $x_k \in \Omega_k = \square_k \cap \Omega$, by convexity of Ω the point $x_{j,k} := \left(x_j + x_k\right)/2$ belongs to Ω. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let $\|x\|_\infty := \max_{1 \le i \le d} |x_i|$ and $B_\infty(x,r) := \left\{y \in \mathbb{R}^d : \|y - x\|_\infty \le r\right\}$. Denote by y_j and y_k the centers of $\square_j$ and $\square_k$ respectively, so that $\square_l = B_\infty(y_l, t/2)$ and $\tilde{\square}_l = B_\infty(y_l, 3t/2)$, $l = j,k$. Since $\square_j$ and $\square_k$ are neighboring cubes, we get that $\left\|y_j - y_k\right\|_\infty = t$. Now, for any point $y \in B_\infty\left(x_{j,k}, t/2\right)$ we obtain

$$\begin{aligned}\left\|y - y_j\right\|_\infty &\le \left\|y - x_{j,k}\right\|_\infty + \left\|y_j - \left(x_j + x_k\right)/2\right\|_\infty \\ &\le \frac{t}{2} + \frac{\left\|y_j - x_j\right\|_\infty}{2} + \frac{\left\|y_j - y_k\right\|_\infty}{2} + \frac{\left\|y_k - x_k\right\|_\infty}{2} \\ &\le \frac{t}{2} + \frac{t}{4} + \frac{t}{2} + \frac{t}{4} = \frac{3t}{2}.\end{aligned}$$

Similarly, $\|y - y_k\|_\infty \le 3t/2$, so that $B_\infty\left(x_{j,k}, t/2\right) \subseteq \tilde{\square}_j \cap \tilde{\square}_k$. Obviously, $B\left(x_{j,k}, t/2\right) \subset B_\infty\left(x_{j,k}, t/2\right)$; so the proof is complete. ◇

Proof of Theorem 1.5. The proof is similar to the proof of Theorem 1.2. For $0 < t \le diam(\Omega)$ let $g_t \in W_p^r(\Omega)$ be defined by (3.6). Then,

$$\begin{aligned}K_r\left(f, t^r\right)_p &\le \|f - g_t\|_p + t^r |g_t|_{r,p} \\ &\le \left(1 - \left(\frac{t}{diam(\Omega)}\right)^r\right) K_r\left(f, t^r\right)_p + \left(\frac{t}{diam(\Omega)}\right)^r \|f - P\|_p \\ &\le C\left(\left(1 - \left(\frac{t}{diam(\Omega)}\right)^r\right) \mu(\Omega, t)^{-(r-1+1/p)} \omega_r(f,t)_p \right. \\ &\quad \left. + \left(\frac{t}{diam(\Omega)}\right)^r \omega_r(f, diam(\Omega))_p\right) \\ &\le C\left(\left(1 - \frac{t^r}{diam(\Omega)^r}\right) \mu(\Omega, t)^{-(r-1+1/p)} + 1\right) \omega_r(f,t)_p. \quad \diamond\end{aligned}$$

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