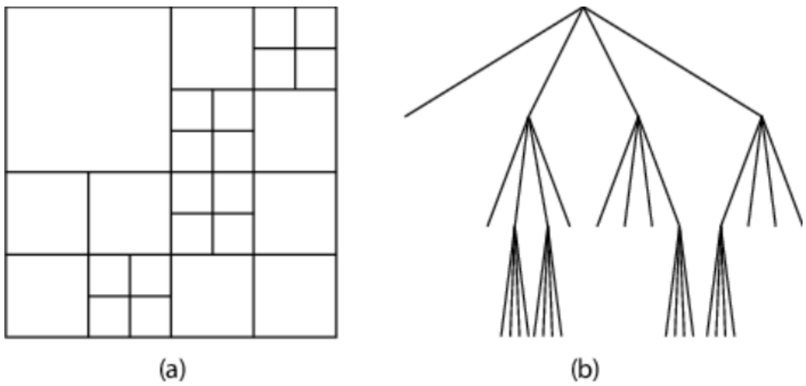


(a) An example Recursive Dyadic Partition (RDP) of a 2-d function domain and (b) associated tree structure.

Theorem Let $f(x) = 1_{\tilde{\Omega}}(x)$, $\tilde{\Omega} \subset [0,1]^n$, a domain with smooth boundary. Then $f \in B_\tau^\alpha(\mathcal{T}_D)$, $\alpha < 1/n\tau$, where $\mathcal{T}_D$ is the non-adaptive tree with partitions at dyadic values along the main axes.

Proof The tree $\mathcal{T}_D$ creates at the level kn domains that are dyadic cubes, with side lengths of 2^{-k}. Any domain $\Omega' \in \mathcal{T}_D$, at level $nk < l < n(k+1)$, is contained in a dyadic cube $\Omega \in \mathcal{T}_D$, at the level nk. From the properties of the modulus of smoothness

$$|\Omega'|^{-\alpha} \omega_r(f,\Omega')_\tau \le 2^{n\alpha} |\Omega|^{-\alpha} \omega_r(f,\Omega)_\tau .$$

Therefore,

$$|f|_{B_\tau^\alpha(\mathcal{T}_D)} = \left(\sum_{\Omega\in\mathcal{T}_D} \left(|\Omega|^{-\alpha} \omega_r(f,\Omega)_\tau \right)^\tau \right)^{1/\tau} \le c \left(\sum_{l(\Omega)=nk, k\ge 0} \left(|\Omega|^{-\alpha} \omega_r(f,\Omega)_\tau \right)^\tau \right)^{1/\tau} .$$

For any $\Omega \in \mathcal{T}_D$, we have that $\omega_r(f,\Omega)_\tau = 0$, if $\partial\tilde{\Omega} \cap \Omega = \emptyset$. Otherwise, if $l(\Omega) = nk$,

$$\omega_r(f,\Omega)_\tau \le \left(\int_\Omega 1^\tau \right)^{1/\tau} = |\Omega|^{1/\tau} = 2^{-kn/\tau} .$$

Therefore,

$$\begin{aligned}
|f|^\tau_{B_\tau^\alpha(\mathcal{T}_D)} &\le c \sum_{l(\Omega)=nk, k\ge 0} \left(|\Omega|^{-\alpha} \omega_r(f,\Omega)_\tau \right)^\tau \\
&\le c \sum_{k=0}^{\infty} \left(2^{kn\alpha} 2^{-kn/\tau} \right)^\tau \#\left\{ \Omega : l(\Omega) = nk,\ \Omega \cap \partial\tilde{\Omega} \ne \emptyset \right\} \\
&= c \sum_{k=0}^{\infty} 2^{kn(\alpha\tau-1)} \#\left\{ \Omega : l(\Omega) = nk,\ \Omega \cap \partial\tilde{\Omega} \ne \emptyset \right\}
\end{aligned}$$

We argue that

$$\#\left\{ \Omega : l(\Omega) = nk,\ \Omega \cap \partial\tilde{\Omega} \ne \emptyset \right\} \le c(\tilde{\Omega}) 2^{k(n-1)} .$$

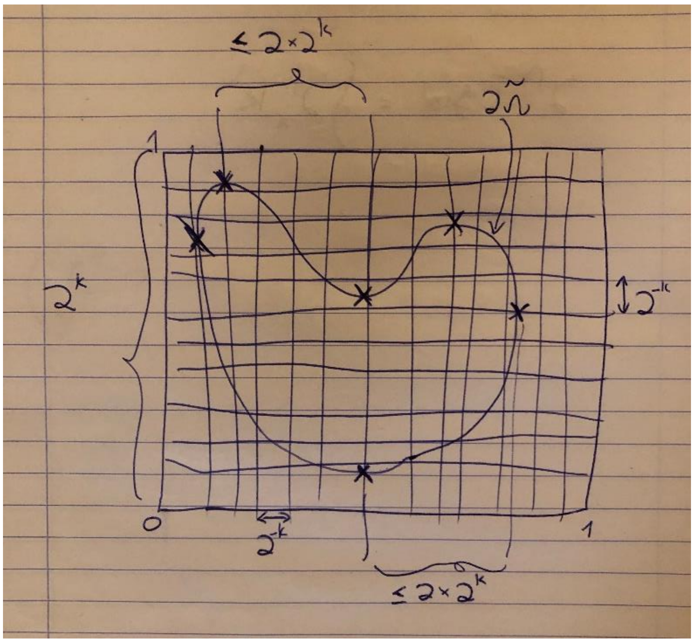
This implies that if $\alpha < 1/n\tau$

$$
\begin{aligned}
|f|^{\tau}_{B^{\alpha}_{\tau}(\mathcal{T}_D)} &\le c\sum_{k=0}^{\infty} 2^{kn(\alpha\tau-1)}\,\#\left\{\Omega : l(\Omega)=nk,\ \Omega\cap\partial\tilde{\Omega}\ne\varnothing\right\} \\
&\le c\sum_{k=0}^{\infty} 2^{kn(\alpha\tau-1)} 2^{k(n-1)} = c\sum_{k=0}^{\infty} 2^{k(n\alpha\tau-1)} < \infty.
\end{aligned}
$$

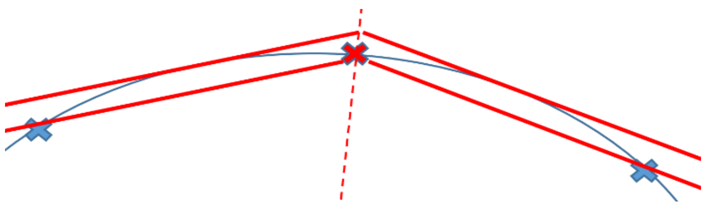
Let's get back to this estimate

$$
\#\left\{\Omega : l(\Omega)=nk,\ \Omega\cap\partial\tilde{\Omega}\ne\varnothing\right\} \le c\left(\tilde{\Omega}\right)2^{k(n-1)}.
$$

Picture argument for $\tilde{\Omega}\subset[0,1]^2$ – There is a finite number of points where the gradient of the boundary of the domain is aligned with one of the main axes. Between these points the boundary segments are monotone in x_1 and x_2 and therefore can only intersect at most 2×2^k dyadic cubes.



Theorem For $n=2$, there exists an adaptive anisotropic tree $\mathcal{T}_A$, such that $f\in B^{\alpha}_{\tau}(\mathcal{T}_A)$, for $\alpha<2/3\tau$ (compared with $\alpha<1/2\tau$ for non-adaptive tree)



Theorem For $n=3$, there exists an adaptive anisotropic tree $\mathcal{T}_A$, such that $f\in B^{\alpha}_{\tau}(\mathcal{T}_A)$, for $\alpha<1/2\tau$ (compared with $\alpha<1/3\tau$ for non-adaptive tree)