Mathematical Foundation of Machine Learning Spring 2020: Assignment II

1. ["Continuous" variance] Let $f:[0,1]^n \to \mathbb{R}^L$ and let $\Omega \subseteq [0,1]^n$. Prove that minimizing the variance over partitions $\Omega' \cup \Omega'' = \Omega$,

$$V_\Omega := \int_{\Omega'} \left| \vec{f}(x) - \vec{E}_{\Omega'} \right|^2_{l_2(\mathbb{R}^L)} dx + \int_{\Omega''} \left| \vec{f}(x) - \vec{E}_{\Omega''} \right|^2_{l_2(\mathbb{R}^L)} dx,$$

is equivalent to maximizing the wavelet norms

$$\left\| \psi_{\Omega'} \right\|^2 + \left\| \psi_{\Omega''} \right\|^2,$$

where $\vec{E}_{\Omega'} = \frac{1}{|\Omega'|} \int_{\Omega'} \vec{f}(x) dx$, $\quad \left\| \psi_{\Omega'} \right\|_2 = |\Omega'|^{1/2} \left| \vec{E}_{\Omega'} - \vec{E}_\Omega \right|_{l_2(\mathbb{R}^L)}$.

Hint: use the method of proof of Lemma 1 in [7].

2. [Convolutions] The convolution of $f, g \in L_1(\mathbb{R}^n)$ is defined by $f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y) dy$.

   (i) Prove that $f * g \in L_1(\mathbb{R}^n)$,

   (ii) Prove that $f * g = g * f$,

   (iii) The Fourier Transform is defined by $\hat{f}(w) = \int_{\mathbb{R}^n} f(x) e^{-i\langle w,x \rangle} dx$, for $w \in \mathbb{R}^n$. Show that

   $$(f * g)^\wedge (w) = \hat{f}(w) \hat{g}(w), \quad \forall w \in \mathbb{R}^n.$$

   (iv) Let $f \in L_1(\mathbb{R}^2)$ be a piecewise constant function. Design a 'filter' $g \in L_\infty(\mathbb{R}^2)$, with support in $[-\varepsilon/2, \varepsilon/2]^2$, for some $\varepsilon > 0$, such that $f * g$ is 'significant' only in ε neighborhoods of points where f has 'almost' vertical edges.