Definition 3.1. For a non-negative self-adjoint operator L on $D(L) \subset L^2(M)$ we define the spectral cutoff operator

$$F_\lambda := \mathbf{1}_{[0,\lambda]}(\sqrt{\mathrm{L}}) = \int_0^\infty \mathbf{1}_{[0,\lambda]}(\sqrt{\mathrm{u}})\mathrm{dE_u}, \qquad \lambda > 0. \tag{3.1}$$

By (1.48), the operator F_λ is bounded on L^2, but not necessarily on L^p, $p \neq 2$. Therefore, before we proceed with the definition of spaces of finite spectra, we note the following. let $\varphi \in C_0^\infty(\mathbb{R})$ be even. Then, by Theorem 2.16 we have that $|\varphi(\sqrt{L})(x,y)| \leq c_k D_{1,k}(x,y)$, for all $k > 2d$ and $x, y \in M$. Therefore, by Proposition (1.12), $\varphi(\sqrt{L})$ is a bounded operator on $L^p(M)$, $1 \leq p \leq \infty$. This supports the following definition

Definition 3.2. Let L be a non-negative self adjoint operator on $L^2(M)$. For any $\lambda > 0$ and $1 \leq p \leq \infty$, we define the **Space of Finite Spectra** or **Spectral Space** as

$$\Sigma_\lambda^p := \{f \in L^p : \quad \varphi(\sqrt{L})f = f, \quad \forall \varphi \in C_0^\infty(\mathbb{R}), \varphi \text{ even and } \varphi \equiv 1 \text{ on } [-\lambda, \lambda]\}.$$

In some cases, we will focus on functions supported in a specific 'band' $[\lambda_1, \lambda_2]$, with $0 < \lambda_1 < \lambda_2 < \infty$ (see e.g. Proposition 4.5). To this end we define

$$\Sigma^p_{[\lambda_1,\lambda_2]} := \{f \in L^p : \quad \varphi(\sqrt{L})f = f, \quad \forall \varphi \in C_0^\infty(\mathbb{R}), \varphi \text{ even and } \varphi \equiv 1 \text{ on } [-\lambda_2, -\lambda_1] \cup [\lambda_1, \lambda_2]\}.$$

At this point, the above equalities $\varphi(\sqrt{L})f = f$ can be understood as the usual equality in the L^p norms, where both functions are in the same equivalence class. We shall now see that the equivalence class contains a continuous representative which will be assumed to be the member of Σ^p_λ we deal with.

PROPOSITION 3.3. For any $f \in L^2$ and $\lambda > 0$, $f \in \Sigma_\lambda^2$ if and only if $F_\lambda f = f$.

PROOF. Assume that for $f \in L^2$, $F_\lambda f = f$. This implies that $f = E_u f$, for any $u \geq \lambda^2$, where $\{E_u\}_{u \geq 0}$ are the orthogonal projections associated with the spectral resolution of L in (1.50). This means that $dE_u f = 0$, $\forall u > \lambda^2$. Now, let $\varphi \in C_0^\infty(\mathbb{R})$, φ even, $\varphi \equiv 1$ on $[-\lambda, \lambda]$. It is easy to see that

$$\begin{aligned}
\varphi(\sqrt{L})f &= \int_0^\infty \varphi(\sqrt{u})dE_u f \\
&= \int_0^{\lambda^2} dE_u f + \int_{\lambda^2}^\infty \varphi(\sqrt{u})dE_u f \\
&= E_{\lambda^2} f + 0 \\
&= f.
\end{aligned}$$

In the other direction, for any $\tilde{\lambda} > \lambda$, construct $\varphi \in C_0^\infty(\mathbb{R})$, φ even ,such that $\varphi \equiv 1$ on $[-\lambda, \lambda]$ and $\mathrm{supp}(\varphi) \subset [-\tilde{\lambda}, \tilde{\lambda}]$. Since $\varphi(\sqrt{L})f = f$, this implies $dE_u f = 0$, for any $u > \tilde{\lambda}^2$. Since $\tilde{\lambda}$ can be chosen arbitrarily close to λ, we get $dE_u f = 0$, for any $u > \lambda^2$. Thus

$$f = \int_0^\infty dE_u f = \int_0^{\lambda^2} dE_u f = F_\lambda f.$$

□

Here are some examples for spaces of finite spectra

(i) Trigonometric polynomials - For the case $M = \mathbb{T}$, $Lf = -f''$ and $\lambda = N$, the operator F_N is the orthogonal projection on the polynomials of degree N, for all $f \in L^2(\mathbb{T})$

$$F_N f(x) = \sum_{k=-\infty}^{\infty} \mathbf{1}_{[0,N]}(|k|)\hat{f}(k)e^{ikx}$$
$$= \sum_{k=-N}^{N} \hat{f}(k)e^{ikx}.$$

This yields the spectral space

$$\Sigma_N^2 = \left\{ T \in L^2(\mathbb{T}) : \quad T(x) = \sum_{k=-N}^{N} a_k e^{ikx} \right\}.$$

(ii) Band-limited functions - For the case $M = \mathbb{R}^d$, $Lf = -\Delta f$ and $\lambda > 0$ we have

$$\Sigma_\lambda^2 = \left\{ f \in L^2(\mathbb{R}^d) : \quad \hat{f}(\omega) = 0, \quad \forall |\omega| > \lambda \right\}.$$

Proposition 3.3. For all $\lambda > 0$ and $1 \le p \le \infty$, one can assume

$$\Sigma_\lambda^p \subset C(M),$$

since each equivalence class has a continuous member. Furthermore, if $p = \infty$, or the non-collapsing condition (1.20) holds, then any $f \in \Sigma_\lambda^p$ is in $\mathrm{Lip}(\alpha)$, with α from Definition 2.1 (see §5.5 for more details on Lipschitz spaces in our setting).

Proof. Let $\theta \in C_0^\infty(\mathbb{R})$, even, with $\mathrm{supp}(\theta) \subset [-2, 2]$, $\theta \equiv 1$ on $[-1, 1]$. By definition, for $f \in \Sigma_\lambda^p$, $f = \theta(\lambda^{-1}\sqrt{L})f$. Let $x \in M$. For any $x' \in M$, $\rho(x, x') \le \lambda^{-1}$, we use (2.30) with $\sigma > 3d/2$, then Hölder's inequality and finally (1.33) to derive for $1 \le p < \infty$

$$\begin{aligned} |\theta(\lambda^{-1}\sqrt{L})f(x) - \theta(\lambda^{-1}\sqrt{L})f(x')| &\le \int_M |\theta(\lambda^{-1}\sqrt{L})(x,y) - \theta(\lambda^{-1}\sqrt{L})(x',y)||f(y)|d\mu(y) \\ &\le c(\lambda\rho(x,x'))^\alpha \int_M D_{\lambda^{-1},\sigma}(x,y)|f(y)|d\mu(y) \\ &\le c|B(x,\lambda^{-1})|^{-1/p}\|f\|_p \lambda^\alpha \rho(x,x')^\alpha. \end{aligned}$$

This certainly shows $\theta(\lambda^{-1}\sqrt{L})f \in C(M)$ and since $f = \theta(\lambda^{-1}\sqrt{L})f$ in L^p, we see the equivalence class has a continuous representative.

Next, for $p = \infty$, using the same method

$$|\theta(\lambda^{-1}\sqrt{L})f(x) - \theta(\lambda^{-1}\sqrt{L})f(x')| \leq \int_M |\theta(\lambda^{-1}\sqrt{L})(x,y) - \theta(\lambda^{-1}\sqrt{L})(x',y)||f(y)|d\mu(y)$$
$$\leq c(\lambda\rho(x,x'))^\alpha \|f\|_\infty \int_M D_{\lambda^{-1},\sigma}(x,y)d\mu(y)$$
$$\leq c\|f\|_\infty \lambda^\alpha \rho(x,x')^\alpha.$$

Also, for $p = \infty$, it is easy to see that if $\rho(x,x') \geq \lambda^{-1}$, using (2.28)

$$|\theta(\lambda^{-1}\sqrt{L})f(x) - \theta(\lambda^{-1}\sqrt{L})f(x')| \leq c\|f\|_\infty$$
$$\leq c\|f\|_\infty \lambda^\alpha \rho(x,x')^\alpha.$$

This shows that for $f \in L^\infty$, $\theta(\lambda^{-1}\sqrt{L})f \in \mathrm{Lip}(\alpha)$.

Assuming the non-collapsing condition (1.20) for $1 \le p < \infty$ gives

$$|B(x, \lambda^{-1})|^{-1/p} \le c\lambda^{d/p}, \quad \forall x \in M,$$

which implies we may proceed for $x, x' \in M$, $\rho(x, x') \le \lambda^{-1}$, with

$$|\theta(\sqrt{L})f(x) - \theta(\sqrt{L})f(x')| \le c\|f\|_p \lambda^{\alpha + d/p} \rho(x, x')^{\alpha}.$$

If $\rho(x, x') \ge \lambda^{-1}$, then using (2.28)

$$\begin{aligned} |\theta(\lambda^{-1}\sqrt{L})f(x) - \theta(\lambda^{-1}\sqrt{L})f(x')| &\le c\lambda^{d/p}\|f\|_p \\ &\le c\|f\|_p \lambda^{\alpha + d/p} \rho(x, x')^{\alpha}. \end{aligned}$$

This again shows $\theta(\lambda^{-1}\sqrt{L})f \in \mathrm{Lip}(\alpha)$. $\square$

PROPOSITION 3.8. Let $\theta \in C_0^\infty(\mathbb{R})$, be even with $\mathrm{supp}(\theta) \subset [-\lambda, \lambda]$. Then, for any $1 \leq p \leq \infty$, $\theta(\sqrt{L})(\cdot, y) \in \Sigma_\lambda^p$, for any $y \in M$ and $\theta(\sqrt{L})(x, \cdot) \in \Sigma_\lambda^p$, for any $x \in M$.

PROOF. First we observe that using (2.28) and (1.33), it is readily seen that for any fixed $y \in M$, $\theta(\sqrt{L})(\cdot, y) \in L^p(M)$, for any $0 < p \leq \infty$. Now let $\varphi \in C_0^\infty(\mathbb{R})$, even, with $\varphi \equiv 1$ on $[-\lambda, \lambda]$. Then, for fixed $y \in M$

$$\begin{aligned}\varphi(\sqrt{L})[\theta(\sqrt{L})(\cdot, y)](x) &= \int_M \varphi(\sqrt{L})(x, z)\theta(\sqrt{L})(z, y)d\mu(z)\\ &= [\varphi(\sqrt{L})\theta(\sqrt{L})](x, y)\\ &= \theta(\sqrt{L})(x, y).\end{aligned}$$

Since $\varphi(\sqrt{L})[\theta(\sqrt{L})(\cdot, y)] = \theta(\sqrt{L})(\cdot, y)$, for any such $\varphi \in C_0^\infty(\mathbb{R})$, we may conclude $\theta(\sqrt{L})(\cdot, y) \in \Sigma_\lambda^p$, for any $y \in M$. Since $\theta(\sqrt{L})$ is self-adjoint with a continuous kernel, by Lemma 2.3, for a fixed $x \in M$, we have that $\theta(\sqrt{L})(x, \cdot) = \overline{\theta(\sqrt{L})(\cdot, x)} \in \Sigma_\lambda^p$. $\square$

One of the famous results for sampling in spaces of finite spectra is the Shannon-Nyquist sampling theorem for band-limited functions in Σ^2_λ, where $M = \mathbb{R}^d$. It states that using the sinc function

$$\varphi(x) = \varphi(x_1, ..., x_d) := \prod_{j=1}^{d} \frac{\sin(\pi x_j)}{\pi x_j},$$

which is the 'ideal' band-limited function with $\hat{\varphi} = \mathbf{1}_{[-\pi,\pi]^d}$, one has

$$f(x) = \sum_{k\in\mathbb{Z}^d} f\left(\frac{\pi}{\lambda}k\right)\varphi\left(\frac{\lambda}{\pi}x - k\right), \quad \forall f \in \Sigma^2_\lambda. \tag{3.22}$$

The Shannon-Nyquist sampling formula implies that band-limited functions can be completely determined from their samples at a frequency which is inversely proportional to their maximal spectra. We can easily derive a generalized Shannon-Nyquist theorem from Theorem 3.17 for smooth cutoff functions

3.3. Maximal δ-nets

In the classical case of band-limited functions over $\mathbb{R}^d$, with maximal spectra λ, there is a notion of critical sampling at the Shannon-Nyquist frequency which is inversely promotional (see (3.22)). Before we proceed to present generalized sampling results on spaces of finite spectra in the next section, we require a general definition of 'critical' sampling, which is the maximal δ-net.

DEFINITION 3.10. For $\delta > 0$, we say that $\mathcal{X} = \{\xi\} \subset M$ is a δ-net, if $\rho(\xi, \eta) \geq \delta$, for all $\xi, \eta \in \mathcal{X}$. We say it is a **maximal δ-net** on M, if it is a δ-net which cannot be enlarged, that is $\rho(x, \mathcal{X}) < \delta$, $\forall x \in M$.

We collect simple properties of δ-nets

Proposition 3.11. Suppose (M, ρ, μ) is a metric measure space obeying the doubling condition (1.14) and let $\delta > 0$.

(i) A maximal δ-net on M always exists.
(ii) If $\mathcal{X}$ is a maximal δ-net on M, then

$$M = \bigcup_{\xi \in \mathcal{X}} B(\xi, \delta), \text{ and } B(\xi, \delta/2) \cap B(\eta, \delta/2) = \emptyset, \quad \forall \xi \neq \eta, \quad \xi, \eta \in \mathcal{X}. \tag{3.5}$$

(iii) A maximal δ-net on M is countable or finite and there exists a disjoint partition $\{A_\xi\}_{\xi \in \mathcal{X}}$ of M consisting of measurable sets such that

$$B(\xi, \delta/2) \subset A_\xi \subset B(\xi, \delta), \quad \xi \in \mathcal{X}. \tag{3.6}$$

Proof. For (i) observe first, that any chain of δ-nets $\mathcal{X}_1 \subset \mathcal{X}_2 \cdots$ has a δ-net upper bound in the form of $\mathcal{X} = \cup_k \mathcal{X}_k$. Next, note that the maximal δ-net we seek is a maximal set in the collection of all δ-nets on M with respect to the natural ordering of sets by inclusion. Hence, we may apply Zorn's lemma to conclude a maximal δ-net on M exists. Property (ii) is easy to see. First, from maximality, for any $x \in M$, there exists $\xi \in \mathcal{X}$, such that $\rho(x,\xi) \leq \delta$. Otherwise, $\mathcal{X}$ can be enlarged using x. Therefore $M = \bigcup_{\xi\in\mathcal{X}} B(\xi,\delta)$. Also, if for some $\xi,\eta \in \mathcal{X}$, $\xi \neq \eta$, we have $B(\xi,\delta/2) \cap B(\eta,\delta/2) \neq \emptyset$, then $\rho(\xi,\eta) \leq \delta$, which is a contradiction.

To prove (iii), we first fix $y \in M$ and observe that for any $n > \delta$, $n \in \mathbb{N}$, using (1.22) and then (1.16), for any $\xi \in \mathcal{X} \cap B(y,n)$, we have $|B(y,n)| \leq c(d,n,\delta)|B(\xi,\delta/2)|$. We use this along with property (ii) to derive

$$\begin{aligned}
\#(\mathcal{X} \cap B(y,n)) \inf_{\eta \in \mathcal{X} \cap B(y,n)} |B(\eta,\delta/2)| &\leq \sum_{\xi \in \mathcal{X} \cap B(y,n)} |B(\xi,\delta/2)| \\
&\leq |B(y,2n)| \\
&\leq c_0 2^d |B(y,n)| \\
&\leq c(d,n,\delta) \inf_{\eta \in \mathcal{X} \cap B(y,n)} |B(\eta,\delta/2)|,
\end{aligned}$$

which yields $\#(\mathcal{X} \cap B(y,n)) \leq c(d,n,\delta) < \infty$, for any $n > \delta$, $n \in \mathbb{N}$. This implies that $\mathcal{X}$ is countable or finite.

We now construct the sets $\{A_\xi\}_{\xi\in\mathcal{X}}$. Let us order the elements of $\mathcal{X}$ in a sequence: $\mathcal{X} = \{\xi_1, \xi_2, ...\}$. We set

$$A_{\xi_1} := B(\xi_1, \delta) \setminus \bigcup_{\eta\in\mathcal{X},\eta\neq\xi_1} B(\eta, \delta/2).$$

It is easy to see that $B(\xi_1, \delta/2) \subset A_{\xi_1} \subset B(\xi_1, \delta)$, since $B(\xi_1, \delta/2)$ does not intersect with any $B(\eta, \delta/2)$, $\eta \neq \xi_1$. In going further, assume $A_{\xi_1}, ..., A_{\xi_{j-1}}$, have been already constructed, we set

$$A_{\xi_j} := B(\xi_j, \delta) \setminus \left[\left(\bigcup_{\nu\leq j-1} A_{\xi_\nu}\right) \bigcup \left(\bigcup_{\eta\in\mathcal{X},\eta\neq\xi_j} B(\eta, \delta/2)\right)\right].$$

Again, it is easy to see that $B(\xi_j, \delta/2) \subset A_{\xi_j} \subset B(\xi_j, \delta)$, since the ball $B(\xi_j, \delta/2)$ was subtracted from all previous sets $A_{\xi_1}, ..., A_{\xi_{j-1}}$ and also at the same time, does not intersect any $B(\eta, \delta/2)$, $\eta \neq \xi_j$. Finally, to see that $M = \bigcup_{\xi\in\mathcal{X}} A_\xi$, observe that $M = \bigcup_{\xi\in\mathcal{X}} B(\xi, \delta)$ and that each A_{xi_j} was constructed by subtracting from $B(\xi_j, \delta)$ subsets of M that are guaranteed to be subsets of A_η, $\eta \neq \xi_j$. $\square$

Lemma 3.12. *Suppose $\mathcal{X}$ is a maximal δ-net on M and that $\{A_\xi\}_{\xi\in\mathcal{X}}$ is a companion disjoint partition of M satisfying* (3.6). *Then the following holds:*

(i) *For any $\sigma > d$ and $\delta_* \ge \delta$*

$$\sum_{\xi\in\mathcal{X}} |A_\xi|(1+\delta_*^{-1}\rho(x,\xi))^{-\sigma} \le c|B(x,\delta_*)|, \quad \forall x \in M. \tag{3.7}$$

(ii) *For any $\sigma > 2d$*

$$\sum_{\xi\in\mathcal{X}} (1+\delta^{-1}\rho(x,\xi))^{-\sigma} \le c, \quad \forall x \in M. \tag{3.8}$$

(iii) *For any $\sigma > 2d$ and $\delta_* \ge \delta$*

$$\sum_{\xi\in\mathcal{X}} |A_\xi| D_{\delta_*,\sigma}(x,\xi) D_{\delta_*,\sigma}(y,\xi) \le c D_{\delta_*,\sigma}(x,y), \quad \forall x, y \in M, \tag{3.9}$$

where $D_{\delta_*,\sigma}$ is defined in (1.29).

PROOF. To prove (i) observe that for $u \in A_\xi \subset B(\xi, \delta) \subset B(\xi, \delta_*)$

$$1 + \delta_*^{-1}\rho(x,u) \le 1 + \delta_*^{-1}\rho(x,\xi) + \delta_*^{-1}\rho(\xi,u) \le 2(1 + \delta_*^{-1}\rho(x,\xi)).$$

Therefore, applying also (1.23)

$$\begin{aligned}\sum_{\xi\in\mathcal{X}} |A_\xi|(1+\delta_*^{-1}\rho(x,\xi))^{-\sigma} &\le c\sum_{\xi\in\mathcal{X}}\int_{A_\xi}(1+\delta_*^{-1}\rho(x,u))^{-\sigma}d\mu(u)\\ &= c\int_M (1+\delta_*^{-1}\rho(x,u))^{-\sigma}d\mu(u)\\ &\le c|B(x,\delta_*)|.\end{aligned}$$

3.4. Sampling Theorems

In this section we study sampling theorem for spectral spaces Σ_λ^p of finite spectra $0 < \lambda < \infty$. These type of results have been studied extensively and have many applications. Let us first recall the classical Marcinkiewicz-Zygmund inequality for trigonometric polynomials on $M = \mathbb{T}$ [**47**]. It states that for any $1 < p < \infty$, there exists a constant $c_p > 0$ such that for any $N \in \mathbb{N}$ with Σ_N as the spectral space of trigonometric polynomials of degree N, the following holds

$$\frac{1}{c_p N} \sum_{j=0}^{N} \left| P\left(\frac{2\pi j}{N+1}\right) \right|^p \leq \|P\|_{L^p(\mathbb{T})}^p \leq \frac{c_p}{N} \sum_{j=0}^{N} \left| P\left(\frac{2\pi j}{N+1}\right) \right|^p, \quad \forall P \in \Sigma_N .$$

THEOREM 3.14. *There exists a constant* $c_b > 0$ *that satisfies the following: Let* $\lambda \geq 1$ *and let* $\mathcal{X}$ *be a maximal* δ*-net on* M *with* $\delta = \gamma/\lambda$, $0 < \gamma \leq 1$. *Suppose* $\{A_\xi\}_{\xi \in \mathcal{X}}$ *is a companion disjoint partition of* M *satisfying* (3.6). *Then for* $\alpha > 0$ *from Definition 2.1 and any* $f \in \Sigma_\lambda^p$, $1 \leq p < \infty$

$$\sum_{\xi \in \mathcal{X}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) \leq c_b^p \gamma^{\alpha p} \|f\|_p^p,$$

and for any $f \in \Sigma_\lambda^\infty$

$$\sup_{\xi \in \mathcal{X}} \sup_{x \in A_\xi} |f(x) - f(\xi)| \leq c_b \gamma^\alpha \|f\|_\infty.$$

Proof. Let φ be an even cutoff function, $\varphi \in C^\infty(\mathbb{R})$, $\operatorname{supp}(\varphi) \subset [-b, b]$, $b > 1$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $[-1, 1]$. Then, for any $f \in \Sigma_\lambda^p$, $f = \varphi(\lambda^{-1}\sqrt{L})f$. Since φ satisfies the conditions of Theorem 2.16, we may use (2.30) with $\sigma > 3d/2$ and $A_\xi \subset B(\xi, \delta)$, $\forall \xi \in \mathcal{X}$, to obtain for $1 \le p < \infty$,

$$
\begin{aligned}
&\sum_{\xi\in\mathcal{X}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) \\
&\quad = \sum_{\xi\in\mathcal{X}} \int_{A_\xi} \left| \int_M (\varphi(\lambda^{-1}\sqrt{L})(x,y) - \varphi(\lambda^{-1}\sqrt{L})(\xi,y)) f(y) d\mu(y) \right|^p d\mu(x) \\
&\quad \le c_1 \sum_{\xi\in\mathcal{X}} \int_{A_\xi} \left(\int_M (\lambda\rho(x,\xi))^\alpha D_{\lambda^{-1},\sigma}(x,y) |f(y)| d\mu(y) \right)^p d\mu(x) \\
&\quad \le c_1 \gamma^{\alpha p} \int_M \left(\int_M D_{\lambda^{-1},\sigma}(x,y) |f(y)| d\mu(y) \right)^p d\mu(x) \\
&\quad \le c_2 \gamma^{\alpha p} \|f\|_p^p,
\end{aligned}
$$

where for the last inequality we used Proposition 1.12. We set $c_b := c_2^{1/p}$. The proof of the case $p = \infty$ is similar. □

Theorem 3.15. [**10**] *Let* $0 < \gamma < 1$ *and assume that*

$$c_b \gamma^\alpha \le \frac{1}{2}, \tag{3.11}$$

where $c_b > 0$ *is from Theorem 3.14. Given* $\lambda \ge 1$, *let* $\mathcal{X}$ *be a maximal* δ*-net on* M *with* $\delta = \gamma/\lambda$. *Suppose* $\{A_\xi\}_{\xi \in \mathcal{X}}$ *is a companion disjoint partition of* M *satisfying* (3.6). *Then, for any* $f \in \Sigma_\lambda^p$, $1 \le p < \infty$

$$\frac{1}{2}\|f\|_p \le \left(\sum_{\xi \in \mathcal{X}} |A_\xi| |f(\xi)|^p \right)^{1/p} \le 2\|f\|_p, \tag{3.12}$$

and for any $f \in \Sigma_\lambda^\infty$

$$\frac{1}{2}\|f\|_\infty \le \sup_{\xi \in \mathcal{X}} |f(\xi)| \le \|f\|_\infty. \tag{3.13}$$

Furthermore, if for $0 < \varepsilon < 1$

$$c_b \gamma^\alpha \leq \frac{\varepsilon}{3}, \tag{3.14}$$

then for any $f \in \Sigma_\lambda^p$, $1 \leq p \leq 2$

$$(1-\varepsilon)\|f\|_p^p \leq \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p \leq (1+\varepsilon)\|f\|_p^p. \tag{3.15}$$

PROOF. We begin with a proof of (3.15). For any $0 < \beta \leq 1$, $a, b \in \mathbb{C}$, and $p \geq 1$ it is easy to see that

$$\frac{1}{(1+\beta)^{p-1}}|a|^p \leq \frac{1}{\beta^{p-1}}|a-b|^p + |b|^p. \tag{3.16}$$

This implies for $1 \leq p \leq 2$

$$(1-\beta)|a|^p \leq \frac{1}{\beta^{p-1}}|a-b|^p + |b|^p. \tag{3.17}$$

The inequality (3.17) with $\beta := \varepsilon/2$ yields for any $f \in L^p$, $1 \leq p \leq 2$, and $\xi \in \mathcal{X}$

$$\Big(1 - \frac{\varepsilon}{2}\Big) \int_{A_\xi} |f(x)|^p d\mu(x) \leq \frac{1}{(\varepsilon/2)^{p-1}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) + |A_\xi||f(\xi)|^p.$$

Summing up over $\xi \in \mathcal{X}$, applying Theorem 3.14 and then using (3.14) gives

$$\begin{aligned}\left(1 - \frac{\varepsilon}{2}\right) \|f\|_p^p &\leq \frac{1}{(\varepsilon/2)^{p-1}} \sum_{\xi \in \mathcal{X}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) + \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p \\ &\leq \frac{1}{(\varepsilon/2)^{p-1}} c_b^p \gamma^{\alpha p} \|f\|_p^p + \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p \\ &\leq \frac{\varepsilon}{2} \|f\|_p^p + \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p.\end{aligned}$$

This proves the left hand side of (3.15). To prove the right hand side of (3.15), we begin by applying (3.17) differently with $\beta = \varepsilon/3$ and any $f \in L^p$, $1 \le p \le 2$, $\xi \in \mathcal{X}$

$$\left(1 - \frac{\varepsilon}{3}\right) |A_\xi| |f(\xi)|^p \le \frac{1}{(\varepsilon/3)^{p-1}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) + \int_{A_\xi} |f(x)|^p d\mu(x).$$

Again, we sum up over $\xi \in \mathcal{X}$, applying Theorem 3.14 and then using (3.14) to get

$$\begin{aligned}
\left(1 - \frac{\varepsilon}{3}\right) \sum_{\xi \in \mathcal{X}} |A_\xi| |f(\xi)|^p &\le \frac{1}{(\varepsilon/3)^{p-1}} \sum_{\xi \in \mathcal{X}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) + \|f\|_p^p \\
&\le \frac{1}{(\varepsilon/3)^{p-1}} c_b^p \gamma^{\alpha p} \|f\|_p^p + \|f\|_p^p \\
&\le \left(1 + \frac{\varepsilon}{3}\right) \|f\|_p^p.
\end{aligned}$$

Since

$$\frac{1+\frac{\varepsilon}{3}}{1-\frac{\varepsilon}{3}} \leq 1+\varepsilon,$$

this proves the right hand side of (3.15).

We turn to the proof of (3.12). By (3.16) with $\beta = 1$, for $1 \leq p < \infty$

$$\frac{1}{2^{p-1}}|a|^p \leq |a-b|^p + |b|^p, \quad \forall a, b \in \mathbb{C}.$$

This implies that for any $\xi \in \mathcal{X}$ we have

$$\frac{1}{2^{p-1}} \int_{A_\xi} |f(x)|^p d\mu(x) \leq \int_{A_\xi} |f(x) - f(\xi)| d\mu(x) + |A_\xi||f(\xi)|^p.$$

Summing up over $\xi \in \mathcal{X}$, applying Theorem 3.14 and now using (3.11)

$$\begin{aligned}
\frac{1}{2^{p-1}} \|f\|_p^p &\leq \sum_{\xi \in \mathcal{X}} \int_{A_\xi} |f(x) - f(\xi)|^p d\mu(x) + \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p \\
&\leq c_b^p \gamma^{\alpha p} \|f\|_p^p + \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p \\
&\leq \frac{1}{2^p} \|f\|_p^p + \sum_{\xi \in \mathcal{X}} |A_\xi||f(\xi)|^p.
\end{aligned}$$

This yields the left hand side of (3.12). Inequality (3.16) with $\beta = 1$ also gives for any $\xi \in \mathcal{X}$

$$\frac{1}{2^{p-1}}|A_\xi||f(\xi)|^p \leq \int_{A_\xi} |f(x) - f(\xi)|d\mu(x) + \int_{A_\xi} |f(x)|^p d\mu(x),$$

which we again sum up over $\xi \in \mathcal{X}$, applying Theorem 3.14 and using (3.11) to drive

$$\begin{aligned}\frac{1}{2^{p-1}} \sum_{\xi\in\mathcal{X}} |A_\xi||f(\xi)|^p &\leq \sum_{\xi\in\mathcal{X}} \int_{A_\xi} |f(x) - f(\xi)|d\mu(x) + \|f\|_p^p \\ &\leq \left(1 + \frac{1}{2^p}\right) \|f\|_p^p.\end{aligned}$$

From this we conclude the right hand side of (3.12). The proof of (3.13) is similar and simpler. □

COROLLARY 3.16. Under the assumptions of Theorem 3.15 we have for any $f \in \Sigma_\lambda^p$, $1 \le p < \infty$

$$c_\gamma \left(\sum_{\xi \in \mathcal{X}} |B(\xi, \lambda^{-1})| |f(\xi)|^p \right)^{1/p} \le \|f\|_p \le 2 \left(\sum_{\xi \in \mathcal{X}} |B(\xi, \lambda^{-1})| |f(\xi)|^p \right)^{1/p}, \tag{3.18}$$

where

$$c_\gamma := \frac{1}{2} c_0^{-1/p} \left(\frac{\gamma}{2} \right)^{d/p}.$$

PROOF. Recall that a maximal δ-net satisfies (3.6)

$$B(\xi,\delta/2) \subset A_\xi \subset B(\xi,\delta), \quad \xi \in \mathcal{X}.$$

This implies by (1.16), with $\delta = \gamma/\lambda$, that for any $\xi \in \mathcal{X}$

$$\begin{aligned} |B(\xi,\lambda^{-1})| &= \left| B\left(\xi, \frac{2}{\gamma}\frac{\delta}{2}\right)\right| \\ &\le c_0 \left(\frac{2}{\gamma}\right)^d \left| B\left(\xi, \frac{\delta}{2}\right)\right| \\ &\le c_0 \left(\frac{2}{\gamma}\right)^d |A_\xi|. \end{aligned}$$

Combining this with the right hand side of (3.12), gives the left hand side of (3.18). To obtain the right hand side of (3.18) we use

$$|A_\xi| \leq |B(\xi, \delta)| = |B(\xi, \gamma\lambda^{-1})| \leq |B(\xi, \lambda^{-1})|, \quad \forall \xi \in \mathcal{X}, 0 < \gamma \leq 1,$$

and the left hand side of (3.12). □

THEOREM 3.17. [**10**] *Given* $\lambda \geq 1$, *let* $\mathcal{X}$ *be a maximal* δ*-net on* M, *with* $\delta = \gamma/\lambda$, $0 < \gamma < 1$ *and*

$$c_b \gamma^\alpha \leq \frac{1}{4}, \tag{3.19}$$

where $c_b > 0$ *is from Theorem 3.14. Then, there exist positive weights* $\{\omega_\xi\}_{\xi \in \mathcal{X}}$ *such that*

$$\int_M f d\mu = \sum_{\xi \in \mathcal{X}} \omega_\xi f(\xi), \quad \forall f \in \Sigma_\lambda^1. \tag{3.20}$$

Furthermore, the weights $\{\omega_\xi\}_{\xi \in \mathcal{X}}$ *satisfy* $\omega_\xi \sim |B(\xi, \delta)|$, *for all* $\xi \in \mathcal{X}$.

Definition 3.19. We say that the spaces of finite spectra $\{\Sigma_\lambda^2\}_{\lambda>0}$ possess the **Polynomial Property** (or **Product Property**), if there exists a constant $\kappa > 1$, such that for any $\lambda > 0$,

$$f, g \in \Sigma_\lambda^2 \Rightarrow fg \in \Sigma_{\kappa\lambda}^2. \tag{3.23}$$

This property is valid for example, for band-limited functions on $\mathbb{R}^d$, or when the spaces of finite spectra are polynomials. Some examples are: trigonometric polynomials, spherical harmonics [**52**], algebraic polynomials on the interval with Jacobi weights [**55**], on the ball [**56**] and in the context of Hermite [**57**] and Laguerre [**44**] expansions. Under this assumption, the following generalizes a result from [**54**] for compact manifolds

Corollary 3.20. Assume the polynomial property with $\kappa \geq 1$. For any $\lambda \geq 1$, $b > 1$, let $\mathcal{X}$ be a maximal δ-net on M, with $\delta = \gamma/(b\kappa\lambda)$, $0 < \gamma < 1$ and

$$c_b \gamma^\alpha \leq \frac{1}{4},$$

where $c_b > 0$ is from Theorem 3.14. Then, there exist positive weights $\{\omega_\xi\}_{\xi \in \mathcal{X}}$, such that for any even cutoff function $\varphi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}(\varphi) \subset [-b, b]$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $[-1, 1]$

$$f(x) = \sum_{\xi \in \mathcal{X}} \omega_\xi \varphi(\lambda^{-1}\sqrt{L})(x, \xi) f(\xi), \quad \forall f \in \Sigma_\lambda^2, \quad \forall x \in M.$$

PROOF. For $f \in \Sigma_\lambda^2$ and a fixed $x \in M$, define $f_x(y) := \varphi(\lambda^{-1}\sqrt{L})(x,y)f(y)$. Since by Proposition 3.8 $\varphi(\lambda^{-1}\sqrt{L})(x,\cdot) \in \Sigma_{b\lambda}^2$, under the polynomial property, it is easy to see that $f_x \in \Sigma_{b\kappa\lambda}^1$. Applying $f = \varphi(\lambda^{-1}\sqrt{L})f$ and then (3.20) we get

$$\begin{aligned} f(x) &= \int_M \varphi(\lambda^{-1}\sqrt{L})(x,y)f(y)d\mu(y) \\ &= \int_M f_x(y)d\mu(y) \\ &= \sum_{\xi\in\mathcal{X}} \omega_\xi f_x(\xi) \\ &= \sum_{\xi\in\mathcal{X}} \omega_\xi \varphi(\lambda^{-1}\sqrt{L})(x,\xi)f(\xi). \end{aligned}$$

□

3.5. The Jackson and Bernstein Estimates for Spaces of Finite Spectra

The purpose of this section is to show generalized versions of two classic estimates from Approximation theory. The Jackson estimate relates to approximation from Σ_λ^p in the $L^p(M)$ norm, $1 \le p \le \infty$, where for $p = \infty$, we consider Uniformly Continuous Bounded functions (UCB). To this end we define

$$\mathcal{E}_\lambda(f)_p := \inf_{g \in \Sigma_\lambda^p} \|f - g\|_p, \quad \lambda \ge 1. \tag{3.23}$$

The approximation from Σ_λ^p is typically coined as linear approximation, since as we shall see in the proof of the Jackson theorem below, it can be efficiently realized using linear operators $T_\lambda : L^p \to \Sigma_\lambda^p$, $\lambda \ge 1$. In the special case of $p = 2$, we obviously have a realization of the optimal approximation using the linear projectors $F_\lambda : L^2 \to \Sigma_\lambda^2$, defined in (3.1), with $\mathcal{E}_\lambda(f)_2 = \|f - F_\lambda(f)\|_2$. In

Proposition 3.20. For any $s > 0$, $\lambda \geq 1$ and $f \in L^2 \cap D(L^{s/2})$, with $L^{s/2} f \in L^2(M)$

$$\mathcal{E}_\lambda(f)_2 = \|f - F_\lambda(f)\|_2 \leq \lambda^{-s} \|L^{s/2} f\|_2. \tag{3.24}$$

Proof. We have using (3.1) and (1.46)

$$\begin{aligned}
\|f - F_\lambda(f)\|_2^2 &= \int_{\lambda^2}^{\infty} d\|E_u f\|_2^2 \\
&= \int_{\lambda^2}^{\infty} u^{-s} u^s d\|E_u f\|_2^2 \\
&\leq \lambda^{-2s} \int_0^{\infty} u^s d\|E_u f\|_2^2 \\
&= \lambda^{-2s} \|L^{s/2} f\|_2^2.
\end{aligned}$$

$$\|\varphi(L) f\|_2^2 = \int_0^{\infty} |\varphi(u)|^2 d\|E_u f\|_2^2.$$

□

THEOREM 3.22. *Let $1 \le p \le \infty$. Then, for any $s > 0$, there exists $c_s > 0$, such that for any $\lambda \ge 1$ and $f \in L^p \cap D(L^{s/2})$, with $L^{s/2} f \in L^p$*

$$\mathcal{E}_\lambda(f)_p \le c_s \lambda^{-s} \|L^{s/2} f\|_p. \tag{3.26}$$

PROOF. Let $\theta \in C^\infty(\mathbb{R})$ be even, $0 \le \theta \le 1$, $\theta \equiv 1$ on $[-1,1]$ and $\operatorname{supp}(\theta) \subseteq [-2,2]$. Set $\varphi := \theta(2^{-1}\cdot) - \theta$. Then $1 - \theta = \sum_{j \ge 0} \varphi(2^{-j}\cdot)$. Given $\lambda > 0$ set $\delta := 2/\lambda$. For any $f \in L^p$, we have that the linear projection $\theta(\delta\sqrt{L})f$ is in Σ^p_λ. Therefore

$$\mathcal{E}_\lambda(f)_p \le \|f - \theta(\delta\sqrt{L})f\|_p \le \sum_{j \ge 0} \|\varphi(2^{-j}\delta\sqrt{L})f\|_p.$$

Denote briefly $h(u) := \varphi(u)|u|^{-s}$, $u \in \mathbb{R}$. Then

$$\varphi(2^{-j}\delta\sqrt{L})L^{-s/2} = (2^{-j}\delta)^s h(2^{-j}\delta\sqrt{L}).$$

We now further assume $L^{s/2}f \in L^p$ and proceed for each $j \geq 0$ with

$$\begin{aligned}\|\varphi(2^{-j}\delta\sqrt{L})f\|_p &= \|\varphi(2^{-j}\delta\sqrt{L})L^{-s/2}L^{s/2}f\|_p \\ &\leq (2^{-j}\delta)^s \|h(2^{-j}\delta\sqrt{L})\|_{L^p\to L^p}\|L^{s/2}f\|_p.\end{aligned}$$

Since for any $s > 0$, the cutoff function h is even, $h \in C^\infty(\mathbb{R})$ and $\mathrm{supp}(h) \subseteq [-4, 4]$, we may apply Theorem 2.16 to derive for $k > 2d$, that

$$|h(2^{-j}\delta\sqrt{L})(x, y)| \leq c_1 D_{2^{-j}\delta,k}(x, y), \quad \forall x, y \in M,$$

where c_1 also depends on s. Therefore, applying Proposition 1.12 gives

$$\|h(2^{-j}\delta\sqrt{L})\|_{L^p\to L^p} \leq c_2,$$

and c_2 also depends on s. We get for any $\lambda \geq 1$

$$\begin{aligned}
\mathcal{E}_\lambda(f)_p &\leq \sum_{j\geq 0} \|\varphi(2^{-j}\delta\sqrt{L})f\|_p \\
&\leq c_2\lambda^{-s}\|L^{s/2}f\|_p \sum_{j\geq 0} 2^{-js} \\
&\leq c_s\lambda^{-s}\|L^{s/2}f\|_p.
\end{aligned}$$

□

Next we proceed with a Bernstein-type estimate for the spectral spaces. We first recall a classic result for trigonometric polynomials on $\mathbb{T}$ [**21**]. Let $\Sigma_N := \{P(x) = \sum_{k=-N}^{N} a_k e^{ikx}\}$. Then for any $r \in \mathbb{N}$, and $P \in \Sigma_N$, $\|P^{(r)}\|_p \leq N^r \|P\|_p$, $1 \leq p \leq \infty$. Additional classic results are for band-limited functions. For example, if $f \in \mathcal{S}(\mathbb{R})$ and $\operatorname{supp}(\hat{f}) \subseteq [-\lambda, \lambda]$, then for any $r \in \mathbb{N}$, $\|f^{(r)}\|_\infty \leq (2\pi)^r \lambda^r \|f\|_\infty$ (see e.g. [**57**, Theorem 2.3.17]).

Corollary 2.19. Suppose $\varphi \in \mathcal{S}(\mathbb{R})$ is even. Then for any $m \in \mathbb{N}$ and $0 < \delta \le 1$, $L^m\varphi(\delta\sqrt{L})$ is an integral operator with a kernel satisfying for any $\sigma > 0$

$$|L^m\varphi(\delta\sqrt{L})(x,y)| \le c\delta^{-2m} D_{\delta,\sigma}(x,y), \quad \forall x, y \in M, \tag{2.38}$$

and the Hölder-type estimate

$$|L^m\varphi(\delta\sqrt{L})(x,y) - L^m\varphi(\delta\sqrt{L})(x,y')| \le c\delta^{-2m}\left(\frac{\rho(y,y')}{\delta}\right)^{\alpha} D_{\delta,\sigma}(x,y), \quad \text{if } \rho(y,y') \le \delta, \tag{2.39}$$

where the constants depend on $c_0, c^\star, C^\star, m$ and the Schwartz constants $\{c(\nu, r)\}$ of φ, for all $0 \le \nu \le \sigma$, $0 \le r \le \sigma + \alpha + d + 1$.

Proof. For $m \ge 0$, let $h(\lambda) := \lambda^{2m}\varphi(\lambda)$ which is even and in $\mathcal{S}(\mathbb{R})$. For any $\delta > 0$, $h(\delta\sqrt{L}) = \delta^{2m}L^m\varphi(\delta\sqrt{L})$. Then the corollary follows readily by applying Theorem 2.18 to h and any selected $\sigma > 0$. □

Theorem 3.22. *Let $1 \le p \le \infty$. Then, for any $r \in \mathbb{N}$, there exists $c_r > 0$, such that for any $\lambda \ge 1$*

$$\|L^r g\|_p \le c_r \lambda^{2r} \|g\|_p, \quad \forall g \in \Sigma_\lambda^p. \tag{3.26}$$

Proof. Choose $\theta \in C^\infty(\mathbb{R}_+)$, even, with $\theta \equiv 1$ on $[-1,1]$ and $\mathrm{supp}(\theta) \subset [-2,2]$. For any $g \in \Sigma_\lambda^p$, we have by definition $g = \theta(\delta\sqrt{L})g$, with $\delta = \lambda^{-1}$. Thus, $L^r g = L^r \theta(\delta\sqrt{L})g$. Then, (3.26) follows by (2.38) and Proposition 1.12

$$\begin{aligned}\|L^r g\|_p &= \|L^r \theta(\delta\sqrt{L})g\|_p \\ &\le \|L^r \theta(\delta\sqrt{L})\|_{L^p \to L^p} \|g\|_p \\ &\le c_r \lambda^{2r} \|g\|_p.\end{aligned}$$

□

Preparation

Let $|M| < \infty$ and $0 < p \leq q \leq \infty$. Then for any $f \in L^q(M)$

$$\|f\|_p \leq |M|^{1/p-1/q}\|f\|_q.$$

For $q = \infty$ we get

$$\|f\|_p = \left(\int_M |f|^p\right)^{1/p} \leq \|f\|_\infty |M|^{1/p}.$$

For $q < \infty$, we use Hölder's inequality for $r = q/p$ and

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

$$\|f\|_p = \left(\int_M |f|^p 1\right)^{1/p} \leq (\||f|^p\|_r \|1\|_{r'})^{1/p} = \left(\int_M |f|^q\right)^{1/q} |M|^{1/p-1/q}.$$

3.2. A Generalized Nikolskii-Type Theorem for Σ_λ^p

It is simple to show that in the case $|M| < \infty$, we have for $0 < p \leq q \leq \infty$ and any $f \in L^q(M)$

$$\|f\|_{L^p(M)} \leq |M|^{1/p-1/q} \|f\|_{L^q(M)}.$$

A Nikolskii-type theorem is, in some sense, an inverse inequality that can be applied to spectral functions in Σ_λ^p, where the spectral radius λ comes into play. Before we present the generalized version, here are some well known special cases.

(i) The theorem for trigonometric polynomials proved by Nikolskii in 1951 [**53**] states that for $1 \leq p \leq q \leq \infty$ and a trigonometric polynomial $T : [-\pi, \pi]^d \to \mathbb{C}$ of the form

$$T(x) = \sum_{j_1=-\lambda_1}^{\lambda_1} \cdots \sum_{j_d=-\lambda_d}^{\lambda_d} a_{j_1,\ldots,j_d} e^{i(j_1 x_1 + \cdots + j_d x_d)},$$

we have

$$\|T\|_{L^q([-\pi,\pi]^d)} \leq 2^d (2\pi)^{d(1/p-1/q)} (\lambda_1 \cdots \lambda_d)^{1/p-1/q} \|T\|_{L^p([-\pi,\pi]^d)}.$$

To compare with the result below, one may use the derived inequality

$$\|T\|_{L^q[-\pi,\pi]^d} \le c \left(\max_{1\le k\le d} \lambda_k \right)^{d(1/p-1/q)} \|T\|_{L^p[-\pi,\pi]^d}.$$

(ii) On $\mathbb{R}^d$ the Nikolskii inequality takes form for $1 \le p \le q \le \infty$ and a band-limited function $g \in L_p(\mathbb{R}^d)$ such that $\operatorname{supp}(\hat{g}) \subseteq [-\lambda, \lambda]^d$

$$\|g\|_{L^q(\mathbb{R}^d)} \le c\lambda^{d(1/p-1/q)} \|g\|_{L^p(\mathbb{R}^d)}.$$

We are ready to present a Nikolskii-type theorem that relates the weighted norms of spectral functions

THEOREM 3.9. [**43**] *Let* $0 < p \le q \le \infty$ *and* $\gamma \in \mathbb{R}$. *Then there exists* $c > 0$, *such that*

$$\||B(\cdot, \lambda^{-1})|^{\gamma} g(\cdot)\|_q \le c \||B(\cdot, \lambda^{-1})|^{\gamma+1/q-1/p} g(\cdot)\|_p, \quad \forall g \in \Sigma_\lambda^*, \quad \lambda \ge 1. \tag{3.2}$$

Assuming further the non-collapsing Condition (1.20), *gives*

$$\|g\|_q \le c\lambda^{d(1/p-1/q)} \|g\|_p, \quad \forall g \in \Sigma_\lambda^*, \quad \lambda \ge 1. \tag{3.3}$$

3.7. Orthogonal Polynomials associated with L in the case $|M| < \infty$

In this section we focus on the case where $|M| < \infty$ (or equivalently, by Theorem 1.5, $\operatorname{diam}(M) < \infty$). We assume the basic setting, where the doubling condition (1.14) holds, L is a non-negative self-adjoint operator and the Gaussian upper bound (2.1) and the weak Markov property (2.45) hold.

THEOREM 3.24. *If $|M| < \infty$ then:*

(i) *For any $\lambda \geq 1$, the spaces of finite spectra Σ_λ^p are equivalent for all $1 \leq p \leq \infty$ and so we may denote them by Σ_λ.*

(ii) *For any $\lambda \geq 1$, Σ_λ are finite dimensional and*

$$\dim(\Sigma_\lambda) \leq c \operatorname{diam}(M)^d \lambda^d.$$

PROOF. To prove (i), we first recall that in the case $|M| < \infty$, for $0 < p \leq q \leq \infty$, we have for any $f \in L^q$

$$\|f\|_p \leq |M|^{1/p-1/q}\|f\|_q,$$

and so $\Sigma_\lambda^q \subseteq \Sigma_\lambda^p$. At the same time, we have by the Nikolskii inequality (3.3) for spaces of finite spectra, under the assumption of the non-collapsing condition (which by (1.17) is satisfied in the case $|M| < \infty$)

$$\|f\|_q \leq c\lambda^{d(1/p-1/q)}\|f\|_p, \quad \forall f \in \Sigma_\lambda^p, \lambda \geq 1.$$

This proves (i).

To prove (ii), we first observe the following. Let $\{E_\lambda\}_{\lambda\geq 0}$ be the orthogonal projections composing the spectral resolution of L such that $L = \int_0^\infty \lambda dE_\lambda$ (see Definition 1.22). Since $\Sigma_\lambda = \Sigma_\lambda^2$, we have that (see Proposition 3.4)

$$f \in \Sigma_{\sqrt{\lambda}} \Leftrightarrow \mathbf{1}_{[0,\sqrt{\lambda}]}(\sqrt{L})f = f \Leftrightarrow E_\lambda f = f.$$

We will show that each E_λ, $\lambda \geq 1$, is an Hilbert-Schmidt operator. Since $E_\lambda = \mathbf{1}_{[0,\sqrt{\lambda}]}(\sqrt{L})$, by Theorem 2.13 it is a continuous kernel operator. Next, since for any $\lambda > 0$, E_λ is a self-adjoint bounded operator on L^2, by Lemma 2.3, $E_\lambda(y,x) = \overline{E_\lambda(x,y)}$, $\forall x,y \in M$. We use this and $E_\lambda^2 = E_\lambda$, to derive for any $x \in M$

$$\int_M |E_\lambda(x,y)|^2 d\mu(y) = \int_M E_\lambda(x,y)E_\lambda(y,x)d\mu(y) = E_\lambda^2(x,x) = E_\lambda(x,x).$$

Recall that by (2.54) we have for any $\lambda \geq 1$

(3.30)

$$\int_M E_\lambda(x,x)d\mu(x) = \int_M \mathbf{1}_{[0,\sqrt{\lambda}]}(\sqrt{L})(x,x)d\mu(x) \sim \int_M |B(x,\lambda^{-1/2})|^{-1}d\mu(x).$$

Hence, using (3.30) and then (1.17), we can bound the Hilbert-Schmidt norm of E_λ

$$\begin{aligned}
\|E_\lambda\|_{HS}^2 &:= \int_M \int_M |E_\lambda(x,y)|^2 d\mu(x)d\mu(y) \\
&= \int_M E_\lambda(x,x)d\mu(x) \\
&\leq c\int_M |B(x,\lambda^{-1/2})|^{-1}d\mu(x) \\
&\leq c\,\mathrm{diam}(M)^d \lambda^{d/2} < \infty.
\end{aligned}$$

Since an Hilbert-Schmidt operator is compact, by the Riesz-Schauder theorem (see [**59**, Theorem 6.15]), the spectrum of E_λ is discrete and any non-zero eigenvalue has finite multiplicity. By the Hilbert-Schmidt theorem (see [**59**, Theorem 6.16]), there exists a countable orthonormal set $\{f_{\lambda,j}\}_{j\in J}$ of eigenfunctions. Observe that in our special case where $E_\lambda^2 f_{\lambda,j} = E_\lambda f_{\lambda,j}$, the only non-zero eigenvalue is 1. Therefore, functional analysis tells us that $\dim(\Sigma_{\sqrt{\lambda}}) = \#J < \infty$. However, here, we can provide a bound for $\#J$ as follows. Obviously the kernel of E_λ is

$$E_\lambda(x, y) = \sum_{j\in J} f_{\lambda,j}(x)\overline{f_{\lambda,j}(y)}.$$

Consequently using the above estimate for $\int_M E_\lambda(x, x)d\mu(x)$

$$\begin{aligned}\dim(\Sigma_{\sqrt{\lambda}}) &= \#J \\ &= \int_M \sum_{j\in J} |f_{\lambda,j}(x)|^2 d\mu(x) \\ &= \int_M E_\lambda(x, x)d\mu(x) \\ &\leq c \operatorname{diam}(M)^d \lambda^{d/2} < \infty.\end{aligned}$$

Theorem 3.25. *If $|M| < \infty$ and $L^2(M)$ is separable, then:*

(i) *The entire spectrum $\sigma(L)$ is the discrete spectrum and consists of an increasing sequence $0 \le \lambda_1 < \lambda_2 < \cdots$. In the case where $\dim(L^2(M)) = \infty$, then $\lim_{j\to\infty} \lambda_j = \infty$.*

(ii) *Denoting $\mathcal{H}_j := \mathrm{Ker}(\lambda_j I - L)$, then $1 \le \dim(\mathcal{H}_j) < \infty$, and $L^2(M) = \oplus_j \mathcal{H}_j$. This further implies that there exists an orthonormal basis of* **polynomial** *eigenfunctions $\{P_{j,l}\}$, where*

$$LP_{j,l} = \lambda_j P_{j,l}, \quad j \ge 1, \quad 1 \le l \le \dim(\mathcal{H}_j),$$

and

$$f = \sum_{j=1}^{\infty} \sum_{l=1}^{\dim(\mathcal{H}_j)} \langle f, P_{j,l} \rangle P_{j,l}, \quad \forall f \in L^2(M).$$