Foundations of Approximation Theory 2020:

Theorem list for the exam

Theorems for 20 points

1. [Hölder Inequality] Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Prove:

   (i) Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \ \forall a, b \ge 0 .$$

   (ii) For $f \in L_p(\Omega), g \in L_{p'}(\Omega)$,

$$\| fg \|_{L_1(\Omega)} \le \| f \|_{L_p(\Omega)} \| g \|_{L_{p'}(\Omega)} .$$

2. [Summability kernel]

   (i) Define the properties of a summability kernel over $\mathbb{T}$.

   (ii) Prove that for a summability kernel $\{h_N\}$ and $f \in C(\mathbb{T})$,

$$\| f - h_N * f \|_{C(\mathbb{T})} = \max_{-\pi \le x \le \pi} \left| f(x) - h_N * f(x) \right| \underset{N \to \infty}{\to} 0 .$$

3. [Piecewise constant approximation of Sobolev functions] Prove that for $g \in W_p^1(\mathbb{R})$, $1 \le p \le \infty$,

$$E\left(g, S(N_1)^h\right)_p \le h \| g' \|_p , \quad h > 0 .$$

4. [Piecewise constant approximation of Lip functions] Prove that for $f \in Lip(\alpha)$, $0 < \alpha < 1$,

$$E_N(f)_{L_\infty([0,1])} := \inf_{\phi \in S(N_1)^{1/N}} \| f - \phi \|_\infty \le C N^{-\alpha} |f|_{Lip(\alpha)} .$$

   Comments:

   (i) You may use the estimate for $g \in C^1[0,1]$, $E_N(g)_\infty \le N^{-1} |g|_{1,\infty}$.

   (ii) You may use the equivalence of the modulus of smoothness and K-functional.

5. [Discrete Besov norm] Define the integral form of the Besov semi-norm over $[0, \infty]$. Prove the equivalency with the discrete dyadic form.

6. [Refinability of B-splines] Show that for $r \ge 1$, the univariate B-spline N_r, satisfies the two-scale relation

$$N_r(x) = \sum_{k=0}^{r} 2^{1-r} \binom{r}{k} N_r(2x - k), \qquad \forall x \in \mathbb{R} .$$

7. [Nikolskii-type equivalence over convex domains] Prove that for any $n, r \ge 1$ and $0 < p, q \le \infty$, there exist constants of equivalence that depend only on these parameters, such that for any bounded convex domain $\Omega \subset \mathbb{R}^n$ and any algebraic polynomial $P \in \Pi_{r-1}\left(\mathbb{R}^n\right)$

$$\|P\|_{L_q(\Omega)} \sim |\Omega|^{1/q-1/p} \|P\|_{L_p(\Omega)}.$$

Comment: You may use John's Theorem and the equivalence of finite dimensional (quasi) Banach spaces.

8. [Projection onto sinc SI spaces] Let $P_{S(\phi)^h}$ be the orthogonal projector onto $S(\phi)^h$, where ϕ is the sinc function. Then for $f \in L_2\left(\mathbb{R}^n\right)$

$$\left(P_{S(\phi)^h} f\right)^{\wedge}(w) = \hat{f}(w) 1_{\left[-h^{-1}\pi, h^{-1}\pi\right]^n}(w), \qquad h > 0.$$

Let $\{\psi_I\}, \{\tilde{\psi}_I\}$, $I = (e, j, k)$, $e \in E$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, be dual Riesz wavelet bases for $L_2\left(\mathbb{R}^n\right)$ where:

(i) $\operatorname{supp}\left(\psi^e\right), \operatorname{supp}\left(\tilde{\psi}^e\right) \subseteq [-M, M]^n$, $e \in E$.

(ii) $\psi^e, \tilde{\psi}^e \in W_\infty^r$, $e \in E$, $r > \alpha$,

(iii) $\psi^e, \tilde{\psi}^e$, $e \in E$, have $r > \alpha$ vanishing moments.

9. Under the above assumptions, Let $F(x) = \sum_{j=1}^{J} c_{I_j} \psi_{I_j}$, where $\left|c_{I_j}\right| \le L$. Prove

$$\|F\|_2 \le C L J^{1/2}.$$

10. [Jackson theorem for Wavelets] Under the above assumptions, let $f \in B_\tau^\alpha\left(\mathbb{R}^n\right)$, $1/\tau = \alpha/n + 1/2$. Denote $\sigma_N(f)_2 := \inf_{g \in \Sigma_N} \|f - g\|_2$, where Σ_N is the collection of N - term (or less) wavelets. Prove that

$$\sigma_N(f)_2 \le c N^{-\alpha/n} |f|_{B_\tau^\alpha}.$$

Comments:

(i) You may use (9).

(ii) You may use the wavelet characterization

$$|f|_{B_\tau^\alpha} \sim \mathcal{N}_\tau(f) := \left(\sum_I \left|\left\langle f, \tilde{\psi}_I \right\rangle\right|^\tau\right)^{1/\tau}.$$

(iii) You may prove the theorem for a series of 'special cases' of N and add a short explanation on how to generalize to any $N \ge 1$.

11. [Bernstein inequality for piecewise polynomials] Let

$$\Sigma_{N,r} := \left\{ \sum_{j=0}^{N-1} P_j \mathbf{1}_{[t_j, t_{j+1})} : T = \{t_j\},\ 0 = t_0 < t_1 < \cdots < t_N = 1,\quad P_j \in \Pi_{r-1} \right\}.$$

Prove that for $\varphi \in \Sigma_{N,r}$, $\frac{1}{\tau} = \alpha + \frac{1}{p}$, $0 < \alpha < r$,

$$|\varphi|_{B_\tau^\alpha} \le C N^\alpha \|\varphi\|_{L_p[0,1]}.$$

Comment: You may use (7).

Theorems for 30 points

12. [Bernstein for trigonometric polynomials]. Prove that for any univariate real trigonometric polynomial of degree N, $T_N \in \Pi_N(\mathbb{T})$:
    - (i) $T_N'(x)^2 + N^2 T_N(x)^2 \le N^2 \|T_N\|_\infty^2,\ \forall x \in \mathbb{T}$.
    - (ii) $\left\|T_N^{(r)}\right\|_\infty \le N^r \|T_N\|_\infty,\ r \ge 1$.

13. [Equivalence of modulus of smoothness K-functional] Let $1 \le p \le \infty$, $r \ge 1$.
    - (i) Prove that for any $g \in W_p^r(\mathbb{R})$, we have $\omega_r(g,t)_p \le t^r |g|_{r,p}$, $t > 0$.
    - (ii) Prove that for any $f \in L_p(\mathbb{R})$, we have $\omega_r(f,t)_p \le c K_r(f,t^r)_p$, $t > 0$.

Comment: You may use the Minkowski integral inequality.

14. [Bramble-Hilbert Lemma for star-shaped domains] Let $\Omega \subset \mathbb{R}^n$ be a bounded star-shaped domain with respect to a ball B of radius ρ and let $\gamma := \operatorname{diam}(\Omega)/\rho$. Prove that for any $g \in C^r(\Omega)$, $r \ge 1$, there exists a polynomial $P \in \Pi_{r-1}(\mathbb{R}^n)$, such that for all $1 \le p < \infty$ and any $0 \le k \le r-1$,
$$|g - P|_{k,p} \le C(n,r)(1+\gamma)^n \operatorname{diam}(\Omega)^{r-k} |g|_{r,k}.$$

Comments: You may use the following:
- (i) The bound on the averaged Taylor remainder,
- (ii) The commutativity of Taylor polynomials and differentiation with respect to affine transforms,
- (iii) The Riesz potential inequality.

15. [Kernel approximation] Assume a kernel operator T, with kernel $K(x,y)$ satisfies for $r \ge 1$
    - (i) $P(x) = TP(x) = \int_{\mathbb{R}^n} K(x,y) P(y)\,dy$, $\forall P \in \Pi_{r-1}(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$.
    - (ii) $|K(x,y)| \le c \dfrac{1}{(1+|x-y|)^{n+r+\varepsilon}}$, for some $\varepsilon > 0$ and any $x, y \in \mathbb{R}^n$.

    Prove that for $f \in C^r(\mathbb{R}^n)$

$$\| f - T_h f \|_\infty \le c h^r |f|_{r,\infty}, \qquad h>0,$$

where

$$T_h f(x) := \int_{\mathbb{R}^n} K_h(x,y) f(y)\,dy, \qquad K_h(x,y) := h^{-n} K\left(h^{-1}x, h^{-1}y\right)$$

Comment: You may use the Taylor remainder estimate

$$R_{r,x} f(y) \le c\,|y-x|^r \max_{z\in B(x,|y-x|)} \max_{|\alpha|=r} \left|\partial^\alpha f(z)\right|.$$

16. [Jackson theorem for trigonometric polynomials]. Prove that for any periodic function $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$, and any $r \ge 1$

$$E_N(f)_p \le C(r)\,\omega_r\left(f, N^{-1}\right)_p,$$

where $E_N(f)_p$ is the degree of approximation by trigonometric polynomials of degree N.

Comment: For the Jackson kernel $J_{N,r}$, you may assume the estimate

$$\int_0^\pi t^k J_{N,r}(t)\,dt \le C(r) N^{-k}, \; k = 0,\dots,2r-2.$$

17. [Spectral approximation order of the sinc] Show that if ϕ is the sinc function, then $\forall r \ge 1$, $\forall f \in W_2^r(\mathbb{R}^n)$,

$$E\left(f, S(\phi)^h\right)_2 \le C(n,r) h^r |f|_{r,2}.$$

Comment: You may use the result of (8).

18. [Jackson & Bernstein machinery] Let the sequence $\Phi := \{\Phi_N\}_{N\ge 0} \subset X$, where X is a Banach space, satisfy

(i) $0 \in \Phi_N, \Phi_0 = 0$,

(ii) $\Phi_N \subset \Phi_{N+1}$,

(iii) $a\Phi_N = \Phi_N$, $\forall a \ne 0$.

(iv) $\Phi_N + \Phi_N \subset \Phi_{cN}$, for some fixed $c > 0$,

(v) $\bigcup_N \Phi_N$ is dense in X,

We denote $E_N(f)_X := \min_{\varphi\in\Phi_N} \| f - \varphi \|_X$. For $r \ge 1$, let $Y = Y_r \subset X$ and assume that the Jackson and Bernstein inequalities hold:

(i) $E_N(g)_X \le c N^{-r} |g|_Y$, $\forall g \in Y$,

(ii) $\left|\varphi\right|_Y \le cN^r \left\|\varphi\right\|_X$, $\forall \varphi \in \Phi_N$.

Then prove the characterization of the approximation space for any $0 < \alpha < r$, $1 \le q < \infty$,

$$A_q^\alpha\left(X\right) = \left(X,Y\right)_{\alpha/r,q} .$$

Comments

(i) You may assume X,Y are Banach spaces (not quasi).

(ii) You may use the discrete form of the semi-norms

$$\left|f\right|_{A_q^\alpha} \sim \left(\sum_{m=0}^{\infty}\left(2^{m\alpha}E_{2^m}\left(f\right)\right)^q\right)^{1/q}, \quad \left|f\right|_{\theta,q} \sim \left(\sum_{m=0}^{\infty}\left(2^{m\theta r}K\left(f,2^{-mr}\right)\right)^q\right)^{1/q} .$$

(iii) You may formulate and use the discrete Hardy inequality.