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Poly-scale refinability and subdivision

S. Dekel [a,b,*] and N. Dyn [a]

[a] *School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel*
[b] *RealTimeImage, 6 Hamasger St., Or-Yehuda, Israel 60408*

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Abstract

A stationary subdivision scheme is a two-scale process, where values at the next level of refinement are computed from the values of the current level using a single given mask $P = \{p_k\}_{k\in\mathbb{Z}^d}$. Under a certain restriction on the mask it can be shown that there exists a distributional solution for the functional equation $\phi = \sum_{k\in\mathbb{Z}^d} p_k\phi(2\cdot -k)$. It is well known that the limit of a convergent subdivision scheme initialized by data $f^0 = \{f_k^0\}_{k\in\mathbb{Z}^d}$ can be represented as $\sum_{k\in\mathbb{Z}^d} f_k^0\phi(x-k)$, where ϕ is a continuous solution of the functional equation. In this work we generalize this framework in the following sense. The (poly) M-scale subdivision scheme computes the next level of refinement from the $M-1$ scales of the previous level, using $M-1$ given masks, $P_m = \{p_{m,k}\}_{k\in\mathbb{Z}^d}$, $m = 1,\dots,M-1$. With a certain restriction on the masks there exists a distributional solution for the poly-scale functional equation $\phi = \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}^d} p_{m,k}\phi(2^m\cdot -k)$. We show that a convergent poly-scale subdivision process initialized by data $f^0 = \{f_k^0\}_{k\in\mathbb{Z}^d}$ converges to $\sum_{k\in\mathbb{Z}^d} f_k^0\phi(x-k)$, where ϕ is a continuous solution of the poly-scale functional equation. In applications, the poly-scale framework allows the design of subdivision schemes with features that are not possible in the standard two-scale case.

Keywords: Subdivision scheme; Refinable function; Refinement equation; Subdivision convergence

1. Introduction

Two-scale subdivision start from a coarse level representation of an object (function, curve, surface)

$$f^0 = \{f_k^0 \in \mathbb{R}^s\colon k \in \mathbb{Z}^d\}, \tag{1.1}$$

(with $s = 1$ for functions, $s = 2$ or 3 for curves, and $s = 3$ for surfaces) and refine repeatedly the representation from the current level to a finer level. This is done by a fixed local rule $f^{j+1} = \mathbb{S}_P f^j$ of the form

$$f_\alpha^{j+1} = \sum_{k\in\mathbb{Z}^d} p_{\alpha-2k} f_k^j, \quad \alpha \in \mathbb{Z}^d, \tag{1.2}$$

* Corresponding author.
E-mail address: shai.dekel@turboimage.com (S. Dekel).

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based on a finite set of non-zero coefficients $P = \{p_k\}_{k\in\mathbb{Z}^d}$ termed the *mask* of the subdivision process. If such a process is convergent, namely for any $f^0 \in l_\infty(\mathbb{Z}^d)$ there exists $f \in C(\mathbb{R}^d)$ satisfying

$$\lim_{j\to\infty} \sup_{k\in\mathbb{Z}^d} \left|\left(\mathbb{S}_P^j f^0\right)_k - f\left(2^{-j}k\right)\right| = 0, \tag{1.3}$$

then the limit function corresponding to the initialization

$$f^0 = \{\delta_{k,0}\}_{k\in\mathbb{Z}^d}, \tag{1.4}$$

satisfies a two-scale refinement equation

$$\phi = \sum_{k\in\mathbb{Z}^d} p_k \phi(2\cdot -k). \tag{1.5}$$

Conversely, a compactly supported solution of the two-scale refinement Eq. (1.5), defines a subdivision scheme via a change in representations in different levels

$$\sum_{k\in\mathbb{Z}^d} f_k \phi(\cdot - k) = \sum_{k\in\mathbb{Z}^d} (\mathbb{S}_P f)_k \phi(2\cdot -k).$$

The analysis of convergence of subdivision schemes and the analysis of smoothness of their limit functions is the subject of many papers, e.g., [5,8–11]. Also, there is a vast literature in the context of wavelet theory about the properties of the solutions of the two-scale refinement Eq. (1.5) (see, e.g., [7] and references therein).

Motivated by the construction of 'optimal' generators of shift invariant spaces in [2] (see also Examples 2.9) we consider *poly-scale* (M-scale) refinement equations each based on several masks $P_m = \{p_{m,k}\}_{k\in\mathbb{Z}^d}$, $m = 1,\dots,M-1$, relating ϕ to its refinements on $M-1$ scales in the following way:

$$\phi = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,k} \phi\left(2^m \cdot -k\right). \tag{1.6}$$

Together with (1.6), a poly-scale (M-scale) subdivision scheme based on the masks $P_1,\dots,P_{M-1}$, is devised so as to yield as a limit function to the initial data (1.4) that is a compactly supported solution of (1.6). In this subdivision scheme, at each level j we have $M-1$ scales denoted by $f^{j,1},\dots,f^{j,M-1}$, where for $j = 0$ we have

$$f^{0,m} = \begin{cases} f^0, & m = 1, \\ 0, & \text{else}. \end{cases}$$

In the $(j+1)$th iteration the following computation is carried

$$f_k^{j+1,m} = \begin{cases} f_k^{j,m+1} + \sum_{\alpha\in\mathbb{Z}^d} p_{m,k-2^m\alpha} f_\alpha^{j,1}, & 1 \leqslant m \leqslant M-2, \\ \sum_{\alpha\in\mathbb{Z}^d} p_{M-1,\, k-2^{M-1}\alpha} f_\alpha^{j,1}, & m = M-1. \end{cases}$$

The limit function of this M-scale subdivision scheme is defined as $f \in C(\mathbb{R}^d)$ satisfying

$$\lim_{j\to\infty} \sup_{k\in\mathbb{Z}^d} \left| f\left(2^{-j}k\right) - \sum_{m=1}^{M-1} f_{2^{m-1}k}^{j,m} \right| = 0.$$

In Section 2, poly-scale refinement equations are investigated. Conditions for the existence of a compactly supported distributional solution of such an equations are derived, together with an explicit form of its Fourier transform. Solutions of poly-scale refinement equations that are related to solutions of two-scale equations and several interesting examples of poly-scale refinable functions are presented.

Section 3 is concerned with poly-scale subdivision schemes. Necessary conditions on the masks for various notions of convergence are derived, together with the relations between the different notions of convergence. Every convergent poly-scale scheme is shown to define a continuous compactly supported solution to the corresponding poly-scale refinement equation. Poly-scale subdivision schemes are presented also as a special case of matrix subdivision and for the univariate case their convergence is analyzed in terms of a factorization of the Laurent polynomials corresponding to the different masks. This tool is further developed for the analysis of the smoothness of the limit functions generated by such schemes. As an example, a family of univariate three-scale subdivision schemes is then analyzed by the above method. For a certain range of parameters these schemes correspond to three-scale refinable functions that are 'almost interpolating,' non-negative, in $C^1(\mathbb{R})$ with support size $2\frac{2}{3}$.

2. Poly-scale refinability

2.1. Preliminaries

We begin by recalling the notion of shift invariant spaces. Let V be a closed subspace of $L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$. We say that V is *shift invariant* (SI) if there exists a set of *generators* Φ such that

$$V = S(\Phi) := \overline{\operatorname{span}}\big\{\phi(\cdot - k) \bigm| \phi \in \Phi,\ k \in \mathbb{Z}^d\big\}.$$

The space V is called a *finite shift invariant* (FSI) space if Φ is a finite set and a *principal shift invariant* (PSI) space if Φ consists of a single element. To approximate functions with arbitrary precision one uses dilates of shift invariant spaces. Let V be an SI space and $h > 0$. We denote by V^h the dilated closed space

$$V^h := \big\{f(\cdot/h) \bigm| f \in V\big\}.$$

Definition 2.1. An SI space $S(\Phi) \subset L_p(\mathbb{R}^d)$ (or its generators Φ) is said to provide L_p-*approximation order* m if for any $h > 0$ and $f \in W_p^m(\mathbb{R}^d)$

$$E\big(f, S(\Phi)^h\big)_p := \inf_{g \in S(\Phi)^h} \|f - g\|_{L_p(\mathbb{R}^d)} \leqslant C h^m |f|_{W_p^m(\mathbb{R}^d)},$$

where $W_p^m(\mathbb{R}^d)$ denotes the Sobolev space of order m equipped with the L_p-norm.

Definition 2.2. A set of tempered distributions $\Phi = \{\phi_1, \ldots, \phi_n\}$ is *two-scale refinable* if there exist matrices $A_k \in M_{n \times n}(\mathbb{R})$, $k \in \mathbb{Z}^d$, such that the following *two-scale relationship* holds

$$\Phi^t = \sum_{k \in \mathbb{Z}^d} A_k \Phi(2 \cdot - k)^t. \tag{2.1}$$

If the distributions Φ are in $L_p(\mathbb{R}^d)$ then we have that $S(\Phi) \subset S(\Phi)^{1/2}$, and $S(\Phi)$ is said to be *refinable*. Using the Fourier transform an equivalent representation for (2.1) is

$$\widehat{\Phi}^t = P\big(2^{-1}\cdot\big)\widehat{\Phi}\big(2^{-1}\cdot\big)^t, \qquad P(w) := 2^{-d} \sum_{k \in \mathbb{Z}^d} A_k e^{-ikw}. \tag{2.2}$$

If the products $P(2^{-1}\cdot) \times \cdots \times P(2^{-N}\cdot)$ converge as $N \to \infty$ then there exists, in the distributional sense, a solution Φ to (2.1) with a Fourier transform

$$\widehat{\Phi} = \lim_{N \to \infty} P\big(2^{-1}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)\widehat{\Phi}\big(2^{-N}\cdot\big)^t = \left(\prod_{j=1}^{\infty} P\big(2^{-j}\cdot\big)\right)\widehat{\Phi}(0)^t. \tag{2.3}$$

In case Φ consists of one function ϕ, then two-scale refinability takes the form

$$\phi = \sum_{k \in \mathbb{Z}^d} p_k \phi(2 \cdot - k),$$

with $p_k \in \mathbb{R}$, $k \in \mathbb{Z}^d$ and we have a representation

$$\hat{\phi} = \left(\prod_{j=1}^{\infty} P\big(2^{-j}\cdot\big)\right)\hat{\phi}(0), \qquad P(w) := 2^{-d} \sum_{k \in \mathbb{Z}^d} p_k e^{-ikw}. \tag{2.4}$$

2.2. The poly-scale functional equation

We now present our generalization of two-scale refinability.

Definition 2.3. A tempered distribution ϕ is *poly-scale* (*M-scale*) *refinable* for some $2 \leqslant M \in \mathbb{N}$ if there exist masks $P_m = \{p_{m,k}\}_{k\in\mathbb{Z}^d}$, $m = 1, \ldots, M-1$, such that the following poly-scale functional equation holds

$$\phi = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,k}\phi\bigl(2^m \cdot -k\bigr). \tag{2.5}$$

If $\phi \in L_p(\mathbb{R}^d)$ we also say that $S(\phi)$ is poly-scale refinable since (2.5) implies the following relation:

$$S(\phi) \subset S(\phi)^{2^{-1}} + \cdots + S(\phi)^{2^{-(M-1)}}. \tag{2.6}$$

The notion of poly-scale refinability can be easily extended to the FSI setting. In this work, for the sake of clarity, we only treat the poly-scale PSI case. We also assume from this point on that the masks $\{P_m\}$ are finitely supported.

Applying the Fourier transform to (2.5) we have

$$\hat{\phi} = \sum_{m=1}^{M-1} P_m\bigl(2^{-m}\cdot\bigr)\hat{\phi}\bigl(2^{-m}\cdot\bigr), \qquad P_m(w) := 2^{-md}\sum_{k\in\mathbb{Z}^d} p_{m,k}e^{-ikw},$$
$$m = 1, \ldots, M-1. \tag{2.7}$$

We now show that the Fourier representation (2.7) yields an infinite product representation similar to (2.4). Let us denote $\widehat{\Phi} := (\hat{\phi}, \hat{\phi}(2^{-1}\cdot), \ldots, \hat{\phi}(2^{-(M-2)}\cdot))$. Assume that $\hat{\phi}(w) = f^{[j]}(w)(\widehat{\Phi}(2^{-j}w))^t$, $j = 0, 1, 2, \ldots,$ where $f^{[j]}(w) := (f^{j,1}(w), \ldots, f^{j,M-1}(w))$ are defined recursively with $f^{[0]}(w) := (1, 0, \ldots, 0)$. Then for $j = 0$ the assumption holds. To get a representation for $j+1$ from the representation for j we replace $\hat{\phi}(2^{-j}w)$ in $\Phi(2^{-j}w)$, using (2.7), and get

$$\begin{aligned}\hat{\phi}(w) &= f^{[j+1]}(w)\bigl(\widehat{\Phi}\bigl(2^{-(j+1)}w\bigr)\bigr)^t\\ &= \bigl(f^{j+1,1}(w), \ldots, f^{j+1,M-1}(w)\bigr)\bigl(\hat{\phi}\bigl(2^{-(j+1)}w\bigr), \ldots,\\ &\qquad \hat{\phi}\bigl(2^{-(j+M-1)}w\bigr)\bigr)^t,\end{aligned}$$

where

$$f^{j+1,m}(w) = \begin{cases} f^{j,m+1}(w) + P_m\bigl(2^{-(j+m)}w\bigr)f^{j,1}(w), & 1 \leqslant m \leqslant M-2,\\ P_{M-1}\bigl(2^{-(j+M-1)}w\bigr)f^{j,1}(w), & m = M-1.\end{cases}$$

We remark in passing that this is exactly the Fourier formulation of a poly-scale subdivision step that is detailed in the next section (see (3.4)). Introducing the matrix:

$$P(w) := \begin{pmatrix} P_1(w) & P_2\bigl(2^{-1}w\bigr) & \cdots & \cdots & P_{M-1}\bigl(2^{-(M-2)}w\bigr)\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & & \vdots & \vdots\\ \vdots & & & 0 & 0\\ 0 & \cdots & 0 & 1 & 0\end{pmatrix}, \tag{2.8}$$

we have

$$f^{[j+1]}(w) = f^{[j]}(w)P\bigl(2^{-(j+1)}w\bigr) = f^{[0]}(w)P\bigl(2^{-1}w\bigr) \times \cdots \times P\bigl(2^{-(j+1)}w\bigr).$$

Thus, formally

$$\begin{aligned}\hat{\phi} &= \lim_{N\to\infty} f^{[N]}(w)\bigl(\widehat{\Phi}\bigl(2^{-N}w\bigr)\bigr)^t\\ &= (1, 0, \ldots, 0)\lim_{N\to\infty} P\bigl(2^{-1}w\bigr) \times \cdots \times P\bigl(2^{-N}w\bigr)\bigl(\hat{\phi}(0), \ldots, \hat{\phi}(0)\bigr)^t.\end{aligned} \tag{2.9}$$

It is interesting to note that the following 'two-scale type' functional equation holds

$$\widehat{\Phi} = \widehat{\Phi}\big(2^{-1}\cdot\big)P\big(2^{-1}\cdot\big)^t, \tag{2.10}$$

where the entries of the matrix P are $2^{M-1}\pi$-periodic functions.

The infinite product representation (2.9) can facilitate the analysis of existence of solutions to (2.5). This is due to the fact that the approach of Heil and Colella [12], designed to handle two-scale matrix refinement equations are general enough to also deal with our poly-scale equations. Some results from [12] were later generalized in [6], using a different approach. Using the methods of proofs of [6] and [12] we obtain the following theorem.

Theorem 2.4. *Let $P_m = \{p_{m,k}\}_{k\in\mathbb{Z}^d}$, $m = 1, \ldots, M-1$, be finitely supported masks and denote*

$$C_m := 2^{-md} \sum_{k\in\mathbb{Z}^d} p_{m,k}, \quad m = 1, \ldots, M-1.$$

If

$$\sum_{m=1}^{M-1} C_m = 1, \tag{2.11}$$

and (for $M \geqslant 3$) the roots of the polynomial

$$q(\lambda) := \lambda^{M-2} + \sum_{m=2}^{M-1}\left(\sum_{j=m}^{M-1} C_m\right)\lambda^{M-m-1}, \tag{2.12}$$

are smaller than 2 in absolute value, then the products

$$P\big(2^{-1}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)(1, \ldots, 1)^t, \tag{2.13}$$

converge uniformly on compact sets and

$$\hat{\phi} = (1, 0, \ldots, 0) \lim_{N\to\infty} P\big(2^{-1}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)(1, \ldots, 1)^t, \tag{2.14}$$

is the Fourier transform of a compactly supported distributional solution of (2.5).

Sketch of proof. Condition (2.11) ensures that $(1, \ldots, 1)$ is an eigenvector of $P(0)$ for the eigenvalue 1, where $P(0)$ is given by (2.8). It is easy to show that all the other eigenvalues of $P(0)$ must be roots of $q(\lambda)$ defined in (2.12). Therefore, our conditions also ensure that the spectral radius of $P(0)$ is less than 2. This allows us to use the method of proof in [6, Theorem 3.2] to show that the infinite products (2.13) converge. Observe that this is a 'restricted' type of convergence. That is, the matrix product might diverge, but when restricted to act on the vector $(1, \ldots, 1)$ it converges. Next we show that $\hat{\phi}$ defined by (2.14) solves the poly-scale Eq. (2.7).

Denoting

$$a^{[N]} := \big(a_1^{[N]}, \ldots, a_{M-1}^{[N]}\big) = P\big(2^{-1}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)(1, \ldots, 1)^t,$$

it is easy to see by (2.8) that we have the relation

$$a_m^{[N]} = a_{m-1}^{[N-1]}\big(2^{-1}\cdot\big), \quad m = 2, \ldots, M-1.$$

Since the infinite product (2.13) converges, the above equality implies

$$\lim_{N\to\infty} P\big(2^{-1}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)(1, \ldots, 1)^t = \big(\hat{\phi}, \ldots, \hat{\phi}\big(2^{-(M-2)}\cdot\big)\big).$$

We can conclude that $\hat{\phi}$ is a solution of (2.7) by

$$\begin{aligned}
\hat{\phi} &= (1, 0, \ldots, 0) \lim_{N\to\infty} P\big(2^{-1}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)(1, \ldots, 1)^t \\
&= (1, 0, \ldots, 0) P\big(2^{-1}\cdot\big) \lim_{N\to\infty} P\big(2^{-2}\cdot\big) \times \cdots \times P\big(2^{-N}\cdot\big)(1, \ldots, 1)^t \\
&= (1, 0, \ldots, 0) P\big(2^{-1}\cdot\big)\big(\hat{\phi}\big(2^{-1}\cdot\big), \ldots, \hat{\phi}\big(2^{-(M-1)}\cdot\big)\big)^t \\
&= \sum_{m=1}^{M-1} P_m\big(2^{-m}\cdot\big)\hat{\phi}\big(2^{-m}\cdot\big).
\end{aligned}$$

Using the method of proofs from [12] one can first show that $\hat{\phi}$ is a tempered distribution and then that its inverse Fourier transform ϕ which solves (2.5) is compactly supported.

Example 2.5. Let us consider the three-scale functional equation

$$\phi = \sum_{k\in\mathbb{Z}^d} p_{1,k}\phi(2\cdot -k) + \sum_{k\in\mathbb{Z}^d} p_{2,k}\phi(4\cdot -k), \tag{2.15}$$

such that P_1, P_2 defined by (2.7) are trigonometric polynomials with coefficients satisfying

$$\sum_{k\in\mathbb{Z}^d} p_{1,2k+\gamma} = C_1, \qquad \sum_{k\in\mathbb{Z}^d} p_{2,4k+\gamma} = C_2, \quad \gamma\in\mathbb{Z}^d,\ C_1+C_2=1. \tag{2.16}$$

In Theorem 3.4 we show that (2.16) is a necessary condition for the strong uniform convergence of the three-scale subdivision scheme associated with the masks P_1, P_2. It is easy to see that for $P(w)$ given in (2.8)

$$P(0) = \begin{pmatrix} C_1 & C_2 \\ 1 & 0 \end{pmatrix},$$

and that $(1,1)$ is an eigenvector corresponding to the eigenvalue 1. The second eigenvalue is $-C_2 = C_1 - 1$. Therefore, Theorem 2.4 implies that for $|C_2| < 2$ or $-1 < C_1 < 3$ there exists a compactly supported distributional solution to (2.5).

If $S(\phi)$ is M-scale refinable, in the sense of (2.6), we can merge the spaces $S(\phi)^{2^{-m}}$, $m = 0,\dots,M-2$, and create a two-scale refinable FSI space $S(\Sigma)$, where

$$\Sigma = \bigcup_{m=0}^{M-2} \Phi_m, \quad \Phi_m := \{\phi(2^m\cdot -r) \mid r\in\{0,\dots,2^m-1\}^d\}. \tag{2.17}$$

Indeed $S(\Sigma)$ is two-scale refinable since

$$s(\Sigma) = S(\Phi) + \cdots + S(\Phi)^{2^{-(M-2)}} \subset S(\Phi)^{2^{-1}} + \cdots + S(\Phi)^{2^{-(M-1)}} = S(\Sigma)^{1/2}. \tag{2.18}$$

This implies the existence of a matrix $\widetilde{P}$, whose entries are 2π-periodic trigonometric polynomials such that

$$\widehat{\Sigma} = \widehat{\Sigma}(2^{-1}\cdot)\widetilde{P}(2^{-1}\cdot)^t. \tag{2.19}$$

Eq. (2.19) is essentially a two-scale vector equation that can be used [13,14] to estimate the smoothness of ϕ. However, there are important cases where Σ in (2.17) fails to be a stable basis of $S(\Sigma)$ (see Theorem 3.13). Moreover, $S(\Sigma)$ may fail to be regular (see [1]). That is, $S(\Sigma)$ might not have a stable basis. Present literature on the analysis of refinable function vectors assumes stability of the generating set (e.g., [6]) or requires stability for an exact computation of the smoothness of the solutions [13,14].

In Section 3.2 we show that the construction (2.17) indicates how to create a two-scale matrix subdivision scheme from a poly-scale scalar subdivision scheme. We show that there are cases where poly-scale subdivision is meaningful and the corresponding two-scale matrix subdivision is not. This is the case with quasi convergent poly-scale schemes (see Definition 3.1).

We conclude this section with the following simple result.

Theorem 2.6. *Let ϕ be a compactly supported solution of a univariate M-scale functional equation of type* (2.5). *Then,*

$$\operatorname{supp}(\phi) \subseteq \left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m-1}\operatorname{supp}(P_m)\right\rangle, \tag{2.20}$$

where $\langle X\rangle$ denotes the convex hull of $X\subset\mathbb{R}$.

Proof. Assume $\mathrm{supp}(\phi) = [a, b]$ and $\mathrm{supp}(P_m) \subseteq [\alpha_m, \beta_m]$, $m = 1, \ldots, M-1$. Using (2.5) it is clear that

$$a \geqslant \min_{1 \leqslant m \leqslant M-1} \left\{ \frac{a + \alpha_m}{2^m} \right\}, \qquad b \leqslant \max_{1 \leqslant m \leqslant M-1} \left\{ \frac{b + \beta_m}{2^m} \right\}.$$

Therefore we obtain

$$a \geqslant \min_{1 \leqslant m \leqslant M-1} \left\{ \frac{\alpha_m}{2^m - 1} \right\}, \qquad b \leqslant \max_{1 \leqslant m \leqslant M-1} \left\{ \frac{\beta_m}{2^m - 1} \right\},$$

which implies (2.20). □

In Section 3.1 we show an alternative approach that recovers (2.20) and also gives the same estimates for the support size in the multivariate case.

2.3. Poly-scale and two-scale relations

It is easy to see that any two-scale refinable function satisfies infinitely many poly-scale functional equations, but it is not true that every poly-scale refinable function is two-scale refinable. The following is a necessary condition for a poly-scale refinable function to have a two-scale relation. While this condition is non-trivial, it uses only the $M-1$ given masks and does not require knowledge of the underlying function.

Proposition 2.7. *Let $\phi \in L_1(\mathbb{R}^d)$ satisfy an M-scale relation* (2.5) *with $M \geqslant 3$. If ϕ is also two-scale refinable with a relation $\hat{\phi} = \tau(2^{-1}\cdot)\hat{\phi}(2^{-1}\cdot)$, with τ a 2π-periodic function, then τ solves the equation*

$$\prod_{m=1}^{M-1} \tau\left(2^{-m} w\right) = \sum_{m=1}^{M-1} \left(\prod_{j=m+1}^{M-1} \tau\left(2^{-j} w\right) \right) P_m\left(2^{-m} w\right). \tag{2.21}$$

Proof. Since ϕ satisfies $\hat{\phi}(w) = \tau(2^{-1}w)\hat{\phi}(2^{-1}w)$ we have that

$$\hat{\phi}\left(2^{-m} w\right) = \hat{\phi}\left(2^{-(M-1)} w\right) \prod_{j=m+1}^{M-1} \tau\left(2^{-j} w\right), \quad m = 0, \ldots, M-2.$$

Substituting into the M-scale relation (2.7) we get

$$\begin{aligned} \hat{\phi}(w) &= \hat{\phi}\left(2^{-(M-1)} w\right) \prod_{j=1}^{M-1} \tau\left(2^{-j} w\right) \\ &= \hat{\phi}\left(2^{-(M-1)} w\right) \sum_{m=1}^{M-1} \left(\prod_{j=m+1}^{M-1} \tau\left(2^{-j} w\right) \right) P_m\left(2^{-m} w\right). \end{aligned}$$

Since $\phi \in L_1(\mathbb{R}^d)$, its Fourier transform is continuous and not identically zero. This implies that we can obtain (2.21) on a compact domain in $\mathbb{R}^d$. Since we assume that all masks are finitely supported, τ and $\{P_m(\cdot)\}$ are trigonometric polynomials and we can conclude that (2.21) holds for all $w \in \mathbb{R}^d$. □

There are examples of functions with good approximation properties that are poly-scale refinable but not two-scale refinable. An important family of such functions is constructed in [2] by differentiating the two-scale refinable B-splines. Here we treat the general case.

Theorem 2.8. *Let ϕ be univariate, compactly supported and two-scale refinable with a corresponding mask P. Let $\varphi = \sum_{n=1}^{M-1} \alpha_n \phi^{(k_n)}$, for some $M \geqslant 2$, where $\phi^{(k)}$ denotes the k-th derivative of a sufficiently smooth ϕ. Then φ is poly-scale refinable.*

Proof. Without loss of generality, we may assume that $\alpha_n \neq 0$, $n = 1, \ldots, M-1$ and that $k_n \neq k_m$ for $n \neq m$, and show that φ is M-scale refinable. For $M = 2$ the proof is trivial. In case $M \geqslant 3$ it is sufficient to prove the existence of trigonometric polynomials $P_m(w)$, $m = 1, \ldots, M-1$, for which

$$\hat{\varphi}(w) = \sum_{m=1}^{M-1} P_m\big(2^{-m}w\big)\hat{\varphi}\big(2^{-m}w\big). \tag{2.22}$$

Assuming the existence of such polynomials we can replace in (2.22) the term $\hat{\varphi}(w)$ by $\hat{\phi}(w)\sum_{n=1}^{M-1}\beta_n w^{k_n}$ with $\beta_n = i^{k_n}\alpha_n$ to obtain

$$\hat{\phi}(w)\sum_{n=1}^{M-1}\beta_n w^{k_n} = \sum_{n=1}^{M-1}\sum_{m=1}^{M-1} 2^{-mk_n}\beta_n w^{k_n} P_m\big(2^{-m}w\big)\hat{\phi}\big(2^{-m}w\big).$$

Denoting $G_m := \prod_{j=m}^{M-1} P(2^{-j}\cdot)$ for $1 \leqslant m \leqslant M-1$ and $G_M = 1$ we can apply the two-scale refinability of ϕ to obtain

$$G_1(w)\sum_{n=1}^{M-1}\beta_n w^{k_n} = \sum_{n=1}^{M-1}\sum_{m=1}^{M-1} 2^{-mk_n}\beta_n w_{k_n} P_m\big(2^{-m}w\big)G_{m+1}(w).$$

Equating coefficients of powers of w in the two sides of this equation we get the set of equations

$$\sum_{m=1}^{M-1} 2^{-mk_n} P_m\big(2^{-m}\cdot\big)G_{m+1} = G_1, \quad n = 1, \ldots, M-1. \tag{2.23}$$

Observe that a solution of (2.23) gives a valid solution of (2.22) for any non-zero coefficients $\{\alpha_n\}$. Denoting $A(w) := (2^{-mk_n}G_{m+1}(w))_{n,m=1,\ldots,M-1}$, we see that $\det(A(w)) = \det(B)\prod_{m=1}^{M-1}G_{m+1}(w)$ with $B = (2^{-mk_n})_{n,m=1,\ldots,M-1}$. Since $k_n \neq k_m$ for $n \neq m$, the matrix B is invertible. Also, as $\prod_{m=1}^{M-1}G_{m+1}(w)$ is a $2^{M-1}\pi$-periodic trigonometric polynomial it vanishes (if at all) on a finite set of points and so $\det(A(w)) \neq 0$ a.e. This implies that a solution to (2.23) exists and is given by $A^{-1}(w)(G_1(w), \ldots, G_1(w))^t$. Finally, to see that a solution of (2.23) defines trigonometric polynomials P_m, $m = 1, \ldots, M-1$ that solve (2.22) observe that

$$A^{-1}(w) = \left(\frac{\mathrm{cof}_{m,n}(A(w))}{\det(A(w))}\right)_{n,m=1,\ldots,M-1} = \left(\frac{\gamma_{n,m}}{G_{n+1}(w)}\right)_{n,m=1,\ldots,M-1},$$

for some coefficient matrix $\{\gamma_{n,m}\}$. We get that for $1 \leqslant n \leqslant M-1$

$$P_n\big(2^{-n}\cdot\big) = \sum_{m=1}^{M-1}\gamma_{n,m}\frac{G_1}{G_{n+1}} = \prod_{j=1}^{n}P\big(2^{-j}\cdot\big)\sum_{m=1}^{M-1}\gamma_{n,m}, \tag{2.24}$$

and P_n is a 2π-periodic trigonometric polynomial. Thus we conclude that φ is M-scale refinable. □

Examples 2.9. The following are examples of poly-scale refinable functions:

1. In [2] (see also [3]) the authors construct the maximal order minimal support (MOMS) family of univariate functions denoted by OM_m, $m \geqslant 1$, each a result of a differential operator acting on N_m, the B-spline of order m. These functions are 'optimal' in the following sense. For any generator ϕ that provides L_2-approximation order m there exists a constant C_ϕ^- such that

$$E\big(f, S(\phi)^h\big)_2 = C_\phi^- h^m + O\big(h^{m+1}\big), \quad h > 0,\ f \in W_2^{m+1}(\mathbb{R}). \tag{2.25}$$

Among all generators that provide approximation order m and have the minimal support size m, the function OM_m minimizes the constant C_ϕ^- in (2.25).

By Theorem 2.8 these functions are poly-scale refinable. For example, for $m = 4$ we have $OM_4 = N_4 + N_4''/42$ which is three-scale refinable. Since $P(w) = (1/16)(1 + e^{-iw})^4$ is the mask of N_4, by virtue of (2.24) the masks of the three-scale relation are

$$P_1(w) = \tfrac{5}{16}\left(1 + e^{-iw}\right)^4, \qquad P_2(w) = -\tfrac{1}{64}\left(1 + e^{-iw}\right)^4\left(1 + e^{-i2w}\right)^4.$$

The authors of [2] make note of the fact that the OM_n functions follow a 'multi-scale difference equation' but do not pursue this topic further. Indeed, their construction of 'optimal' functions that are M-scale refinable for $M \geqslant 3$ is the main motivation for our work.

2. It is easy to see that for each $n \in N$, the step function

$$\phi_n(x) = \begin{cases} 1, & x \in [-2n/3, 2n/3], \\ 0, & \text{else}, \end{cases}$$

is three-scale refinable with

$$\phi_n(x) = \phi_n(2x) + \phi_n(4x + 2n) + \phi_n(4x - 2n).$$

Next we show how from any poly-scale refinable function we can construct a multitude of poly-scale refinable functions by convolutions with two-scale refinable functions.

Theorem 2.10. *Let $\phi \in L_1(\mathbb{R}^d)$ be M-scale refinable with masks $\{Q_m\}_{m=1}^{M-1}$ and let $\rho \in L_1(\mathbb{R}^d)$ be two-scale refinable with a corresponding mask P. Then $\phi * \rho$ is M-scale refinable with masks*

$$P_m(w) = Q_m(w) \prod_{n=1}^{m} P\left(2^{m-n} w\right), \quad m = 1, \ldots, M-1.$$

Proof. The proof is a direct consequence of the property $\widehat{\phi * \rho} = \hat{\phi}\hat{\rho}$

$$\begin{aligned} \widehat{\phi * \rho}(w) &= \sum_{m=1}^{M-1} Q_m\left(2^{-m} w\right)\left(\prod_{n=1}^{m} P\left(2^{-n} w\right)\right)\hat{\phi}\left(2^{-m} w\right)\hat{\rho}\left(2^{-m} w\right) \\ &= \sum_{m=1}^{M-1} Q_m\left(2^{-m} w\right)\left(\prod_{n=1}^{m} P\left(2^{-n} w\right)\right)\widehat{\phi * \rho}\left(2^{-m} w\right). \qquad \square \end{aligned}$$

Corollary 2.11. *Let $\phi \in L_1(\mathbb{R})$ be M-scale refinable with masks $\{Q_m\}_{m=1}^{M-1}$. Then the function $\phi * N_r$ is M-scale refinable with masks*

$$P_m(w) = Q_m(w) \prod_{n=1}^{m} P\left(2^{m-n} w\right), \qquad P(w) = \left(\frac{1 + e^{-iw}}{2}\right)^r, \quad m = 1, \ldots, M-1. \tag{2.26}$$

Furthermore, $\phi * N_r$ is in C^{r-1} and provides approximation order r. If ϕ is also known to be continuous then we have that $\phi * N_r \in C^r$. This will become useful in Section 3.3 *where we analyze the smoothness of functions generated by poly-scale subdivision.*

We can also show the reverse direction.

Theorem 2.12. *Let $\{Q_m\}_{m=1}^{M-1}$ be masks for which the restricted infinite product* (2.14) *converges to $\hat{\phi}$. Let T, $T(0) = 1$, be a mask for which the infinite product* (2.4) *converges to $\hat{\rho}$. Then the restricted infinite product* (2.14) *defined by the masks*

$$P_m(w) = Q_m(w) \prod_{n=1}^{m} T\left(2^{m-n} w\right),$$

converges to $\phi * \rho$ which is a distributional solution of (2.7).

Sketch of proof. For $N \geqslant 1$ we denote

$$\bigl(a_1^{[N]}(w), \dots, a_{M-1}^{[N]}(w)\bigr) := (1,0,\dots,0)Q\bigl(2^{-1}w\bigr) \times \cdots \times Q\bigl(2^{-N}w\bigr) \quad \text{and}$$
$$\bigl(b_1^{[N]}(w), \dots, b_{M-1}^{[N]}(w)\bigr) := (1,0,\dots,0)P\bigl(2^{-1}w\bigr) \times \cdots \times P\bigl(2^{-N}w\bigr),$$

where $P(w)$ is the matrix as in (2.8) with respect to $\{P_m\}$ and $Q(w)$ is the matrix defined by (2.8) with respect to $\{Q_m\}$. One can show, by induction, that

$$\bigl(b_1^{[N]}(w), \dots, b_{M-1}^{[N]}(w)\bigr) = \prod_{j=1}^{N} T\bigl(2^{-j}w\bigr)\Biggl(a_1^{[N]}(w), a_2^{[N]}(w)T\bigl(2^{-(N+1)}w\bigr), \dots, a_{M-1}^{[N]}(w)\prod_{m=1}^{M-2} T\bigl(2^{-(N+m)}w\bigr)\Biggr). \tag{2.27}$$

If (2.27) holds then the theorem is proved since

$$\begin{aligned}
&(1,0,\dots,0)\lim_{N\to\infty} P\bigl(2^{-1}w\bigr) \times \cdots \times P\bigl(2^{-N}w\bigr)(1,\dots,1)^t \\
&\quad = \lim_{N\to\infty}\sum_{m=1}^{M-1} b_m^{[N]}(w) \\
&\quad = \lim_{N\to\infty}\prod_{j=1}^{N} T\bigl(2^{-j}w\bigr)\lim_{N\to\infty}\sum_{m=1}^{M-1}\Biggl(\prod_{n=1}^{m-1} T\bigl(2^{-(N+n)}w\bigr)\Biggr)a_m^{[N]}(w) \\
&\quad = \hat{\rho}(w)\lim_{N\to\infty}\sum_{m=1}^{M-1} a_m^{[N]}(w) = \hat{\rho}(w)\hat{\phi}(w). \qquad \Box
\end{aligned}$$

3. Poly-scale subdivision

The generalization of two-scale refinability to poly-scale refinability naturally leads to a generalization of classical subdivision theory. As we shall see, the fact that poly-scale refinable spaces can be reorganized as a two-scale refinable FSI (see the construction (2.17)) means that we can formulate poly-scale subdivision as a special case of matrix subdivision. However, during this embedding we lose the structure presented below.

3.1. Poly-scale subdivision and its poly-scale refinable function

The poly-scale subdivision can be defined in terms of 'a change of representation.' Suppose

$$f = \sum_{k\in\mathbb{Z}^d} f_k^0 \phi(\cdot - k), \tag{3.1}$$

and ϕ is M-scale refinable

$$\phi = \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}^d} p_{m,k}\phi\bigl(2^m \cdot - k\bigr). \tag{3.2}$$

Then, substituting ϕ from (3.2) into (3.1) we get

$$f = \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}^d} f_k^{1,m}\phi\bigl(2^m \cdot - k\bigr),$$
$$\text{with } f_k^{1,m} = \sum_{\alpha\in\mathbb{Z}^d} p_{m,k-2^m\alpha} f_\alpha^0,\ k\in\mathbb{Z}^d,\ m = 1,\dots,M-1.$$

In general, with f given in the form

$$f = \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} f_k^{j,m} \phi\left(2^{j+m-1} \cdot -k\right), \tag{3.3}$$

we can use (3.2) to substitute for $\phi(2^j \cdot -k)$ and get

$$f = \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} f_k^{j+1,m} \phi\left(2^{j+m} \cdot -k\right),$$

$$\text{with } f_k^{j+1,m} = \begin{cases} f_k^{j,m+1} + \sum_{\alpha \in \mathbb{Z}^d} p_{m,k-2^m\alpha} f_\alpha^{j,1}, & 1 \leqslant m \leqslant M-2, \\ \sum_{\alpha \in \mathbb{Z}^d} p_{M-1,k-2^{M-1}\alpha} f_\alpha^{j,1}, & m = M-1. \end{cases} \tag{3.4}$$

Thus the poly-scale subdivision scheme maps $M-1$ sequences of coefficients (control points) at level j

$$f^{j,m} := \left\{f_k^{j,m}\right\}_{k \in \mathbb{Z}^d}, \quad m = 1, \ldots, M-1,$$

to the corresponding $M-1$ sequences at level $j+1$ according to (3.4). We term $f^{j,m}$ the control points at the mth scale of the jth level. The limit function of the subdivision scheme is the limit of the piecewise linear functions

$$F^j(x) := \sum_{m=1}^{M-1} \sum_{k \in \mathbb{Z}^d} f_k^{j,m} H\left(2^{j+m-1}x - k\right), \tag{3.5}$$

with H the 'hat function'

$$H_1(x) := \begin{cases} 1 - |x|, & |x| \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.1. An M-scale scheme $\mathbb{S}$ is *strongly convergent* if for any initial data $f^0 \in l_\infty(\mathbb{Z}^d)$ the $M-1$ sequences of functions

$$\left\{\sum_{k \in \mathbb{Z}^d} f_k^{j,m} H\left(2^{j+m-1}x - k\right)\colon\ j \geqslant 1\right\}, \quad m = 1, \ldots, M-1,$$

are uniformly convergent or equivalently there exist continuous functions $f_1, \ldots, f_{M-1}$ such that

$$\lim_{j \to \infty} \sup_{k \in \mathbb{Z}^d} \left| f_k^{j,m} - f_m\left(2^{-(j+m-1)}k\right)\right| = 0, \quad m = 1, \ldots, M-1. \tag{3.6}$$

We also require that there exists some initial data f^0 for which the function f_1 is not identically zero. Under the condition of strong convergence, the values (control points)

$$F^j := \left\{F_k^j := \sum_{m=1}^{M-1} f_{2^{m-1}k}^{j,m}\colon\ k \in \mathbb{Z}^d\right\}, \tag{3.7}$$

tend with $j \to \infty$ uniformly to $f = \sum_{m=1}^{M-1} f_m$. If only this last convergence holds then we term the scheme as *quasi convergent*.

Remark. Observe that for the case of two-scale subdivision ($M = 2$) the notions of strong and quasi convergence are identical. Examples of quasi convergent schemes that are not strongly convergent are presented in Section 3.3.

Next we describe an alternative (perhaps simpler) form of the poly-scale subdivision process that is useful. Assume $\mathbb{S}$ is an M-scale subdivision scheme, f^0 is the initial data and set $f^{-M+2} = \cdots = f^{-1} = 0$ for $M \geqslant 3$. Then (3.4) implies that $f^j := f^{j,1}$, $j \geqslant 1$, can by computed from the first scales of the previous $M-1$ levels by

$$f_k^j = \sum_{m=1}^{M-1} \sum_{\alpha \in \mathbb{Z}^d} p_{m,k-2^m\alpha} f_\alpha^{j-m}, \quad k \in \mathbb{Z}^d. \tag{3.8}$$

Although in this form of poly-scale subdivision, at each level j, only the first scale is computed, we can obtain using (3.4) the values of all the $M-1$ scales by

$$f_k^{j,m} = \sum_{r=m}^{M-1} \sum_{\alpha\in\mathbb{Z}^d} p_{r,k-2^r\alpha} f_\alpha^{j+m-r-1}, \quad m=1,\dots,M-1. \tag{3.9}$$

Definition 3.2. Let $\mathbb{S}$ be an M-scale subdivision scheme. We denote by $\widetilde{\mathbb{S}}$ the scheme that generates from any initial data f^0 at each level j only the first scale $f^j := f^{j,1}$, using (3.8), with $f^{-M+2} = \cdots = f^{-1} = 0$ for $M \geqslant 3$. We call $\widetilde{\mathbb{S}}$ the *partial scheme* of $\mathbb{S}$. We say that $\widetilde{\mathbb{S}}$ is *convergent* if for any initial data $f^0 \in l_\infty(\mathbb{Z}^d)$ the sequence $\{f^j\}_{j\geqslant 1}$ converges, namely

$$\lim_{j\to\infty} \sup_{k\in\mathbb{Z}^d} \left| f_k^j - f\left(2^{-j}k\right)\right| = 0,$$

for some continuous function f. In case $\widetilde{\mathbb{S}}$ is convergent we say that $\mathbb{S}$ is *partially strongly convergent*.

Note 3.3. Let $m \geqslant 1$ and $d \geqslant 1$. We denote $E_m^d := \{0,\dots,2^m-1\}^d$. For a vector $\gamma = (\gamma_1,\dots,\gamma_d) \in \mathbb{Z}^d$, we denote

$$\gamma\left(\bmod 2^m\right) := \left(\gamma_1\left(\bmod 2^m\right),\dots,\gamma_d\left(\bmod 2^m\right)\right).$$

Theorem 3.4. *Let $\mathbb{S}$ be an M-scale subdivision scheme with masks $P_m = \{p_{m,k}\}$, $m = 1,\dots,M-1$. If $\mathbb{S}$ is strongly convergent then there exist constants C_m, $m=1,\dots,M-1$ such that*

$$\begin{aligned} &1.\quad \textstyle\sum_{k\in\mathbb{Z}^d} p_{m,2^m k+\gamma} = C_m, \ \textit{for all}\ \gamma\in E_m^d,\\ &2.\quad \textstyle\sum_{m=1}^{M-1} C_m = 1. \end{aligned} \tag{3.10}$$

Proof. By definition of a strongly convergent scheme, there exists initial data f^0 such that $\mathbb{S}^\infty f^0 = \sum_{m=1}^{M-1} f_m$ with $f_m \in C(\mathbb{R}^d)$ and for some dyadic point $2^{-j_0}k_0$, $f_1(2^{-j_0}k_0) \neq 0$. We now use the partial scheme $\widetilde{\mathbb{S}}$ which generates only the sequences $f^j := f^{j,1}$. Using (3.8) we have for $j > j_0 + M$ and any $\gamma \in E_{M-1}^d$

$$f^j_{2^{j-j_0}k_0+\gamma} = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,\gamma-2^m k} f^{j-m}_{k+2^{j-j_0-m}k_0}. \tag{3.11}$$

Since the masks $\{P_m\}$ have finite support and f^j converges uniformly to $f_1 \in C(\mathbb{R}^d)$, we have for j large enough and for $\alpha = k + 2^{j-j_0-m}k_0$, with finite k, $f_\alpha^{j-m} \approx f_1(2^{-j_0}k_0) \neq 0$, $m = 0,\dots,M-1$. Thus, we can derive from (3.11)

$$\left| 1 - \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,\gamma-2^m k}\right| \leqslant A\varepsilon, \quad \gamma \in E_{M-1}^d,$$

where A is a constant which depends on the sum of the sizes of the supports of $\{P_m\}$ and $\varepsilon > 0$ can be made arbitrarily small by increasing j. We conclude that

$$\sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,2^m k+\gamma} = 1, \quad \gamma \in E_{M-1}^d.$$

It remains to show that

$$\sum_{k\in\mathbb{Z}^d} p_{m,2^m k+\gamma_m} \equiv C_m, \quad m=1,\dots,M-1,\ \gamma_m \in E_m^d.$$

We now use the scheme $\mathbb{S}$. Observe that for $M = 2$ we are done and so we can assume that $M \geqslant 3$. For any $1 \leqslant m \leqslant M-2$ and $j \geqslant j_0 + M$ we have by (3.4)

$$f_k^{j,m} = f_k^{j-1,m+1} + \sum_{\alpha\in\mathbb{Z}^d} p_{m,k-2^m\alpha} f_\alpha^{j-1,1}.$$

Let $\alpha = 2^{j+m-j_0}k_0 + \gamma_m$ with $\gamma_m \in E_m^d$. Observe that $\alpha \equiv \gamma_m (\mathrm{mod}\ 2^m)$. Since the scheme is strongly convergent, this implies that for large enough $j > j_0$ we have locally about the point $2^{-j_0}k_0$

$$f_m\bigl(2^{-j_0}k_0\bigr) \approx f_{m+1}\bigl(2^{-j_0}k_0\bigr) + f_1\bigl(2^{-j_0}k_0\bigr) \sum_{\alpha\in\mathbb{Z}^d} p_{m,2^m\alpha+\gamma_m}.$$

Thus, since $f_1(2^{-j_0}k_0) \neq 0$ we obtain for $1 \leqslant m \leqslant M-2$

$$\sum_{\alpha\in\mathbb{Z}^d} p_{m,2^m\alpha+\gamma_m} = \frac{f_m - f_{m+1}}{f_1}\bigl(2^{-j_0}k_0\bigr) := C_m, \quad \gamma_m \in E_m^d.$$

Together with the first part of the proof, this implies that $\sum_{k\in\mathbb{Z}^d} p_{M-1,2^{M-1}k+\gamma} = C_{M-1}$, $\gamma \in E_{M-1}^d$ and that $\sum_{m=1}^{M-1} C_m = 1$. □

Theorem 3.5. *Assume the masks $\{P_m\}_{m=1}^{M-1}$ of a quasi convergent M-scale scheme meet the following conditions (up to a shift)*

1. $p_{m,2^m k} = 0$ *for* $k \neq 0$ *and* $m = 1, \ldots, M-1$,
2. $\sum_{m=1}^{M-1} p_{m,0} = 1$,

then the scheme is interpolatory in the sense that $\mathbb{S}^\infty f^0(\alpha) = f_\alpha^0$, $\alpha \in \mathbb{Z}^d$.

Proof. We show that the above conditions ensure that for any $j \geqslant 0$ and $\alpha \in \mathbb{Z}^d$

$$F_{2\alpha}^{j+1} = F_\alpha^j, \quad \alpha \in \mathbb{Z}^d,\ j \geqslant 0. \tag{3.12}$$

The proof of (3.12) is by direct computation

$$\begin{aligned}
F_{2\alpha}^{j+1} &= \sum_{m=1}^{M-1} f_{2^m\alpha}^{j+1,m} \\
&= \sum_{m=1}^{M-2} f_{2^m\alpha}^{j,m+1} + \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,2^m(\alpha-k)} f_k^{j,1} \\
&= \sum_{m=1}^{M-2} f_{2^m\alpha}^{j,m+1} + \sum_{m=1}^{M-1} p_{m,0} f_\alpha^{j,1} \\
&= \sum_{m=1}^{M-2} f_{2^m\alpha}^{j,m+1} + f_\alpha^{j,1} \\
&= \sum_{m=0}^{M-2} f_{2^m\alpha}^{j,m+1} = F_\alpha^j.
\end{aligned}$$

Since $F_{2^j\alpha}^j = F_\alpha^0 = f_\alpha^0$ it follows that $\mathbb{S}^\infty f^0(\alpha) = \lim_{j\to\infty} F_{2^j\alpha}^j = f_\alpha^0$. □

Examples 3.6.

1. It is easy to see that for $M = 2$, the necessary conditions (3.10) recover the two-scale subdivision necessary conditions for uniform convergence
$$\sum_{k\in\mathbb{Z}^d} p_{1,2k+\gamma} = 1, \quad \gamma \in E_1^d.$$
2. Let us define a family $\{\mathbb{S}_\beta | \beta \in \mathbb{R}\}$ of univariate three-scale subdivision schemes. Each member of $\{\mathbb{S}_\beta\}$ is defined by its two masks
$$P_{\beta,1}(w) = \frac{\beta}{2} e^{-iw}\bigl(1 + e^{iw}\bigr)^2,$$
$$P_{\beta,2}(w) = \frac{1-\beta}{16} e^{-4iw}\bigl(1 + e^{iw}\bigr)^4\bigl(1 + e^{2iw}\bigr)^2.$$

Observe that for the choice $\beta = 1$, we obtain the two-scale scheme corresponding to the linear B-spline N_2. The family $\{\mathbb{S}_\beta\}$ has the following properties:

a. The schemes $\{\mathbb{S}_\beta\}$ satisfy the necessary conditions (3.10) with $C_1 = \beta$, $C_2 = 1 - \beta$.
b. We prove in Example 3.23 that for the range $-1/3 < \beta < 1$, the scheme is quasi convergent and C^1. For instance, the scheme is strongly convergent and smooth for the choice $\beta = 1/2$. In this case the masks are

$$P_{1/2,1} = \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}\right\}, \qquad P_{1/2,2} = \left\{\tfrac{1}{32}, \tfrac{4}{32}, \tfrac{8}{32}, \tfrac{12}{32}, \tfrac{14}{32}, \tfrac{12}{32}, \tfrac{8}{32}, \tfrac{4}{32}, \tfrac{1}{32}\right\}.$$

   The scheme $\mathbb{S}_{1/2}$ also has the following features: it is 'almost interpolating,' it is shape preserving and reproduces polynomials of degree one.
c. For a quasi convergent scheme $\mathbb{S} \in \{\mathbb{S}_\beta\}$ the corresponding $\mathbb{S}$-refinable function (see Theorem 3.9) has support in $[-4/3, 4/3]$. Recall that in the special case of two-scale subdivision, there are known sharp bounds on the smoothness of the scheme in terms of the support size of the mask (see [5, Corollary 2.1] and [8, Theorem 5.1]). In this sense of maximal smoothness for minimal support size, the schemes $\{\mathbb{S}_\beta\}$ are optimal for the range $-1/3 < \beta < 1$.

Next we show that the requirement in the definition of a strongly convergent scheme for scale-wise convergence tightly couples the continuous components $\{f_m\}_{m=1,\ldots,M-1}$ of the limit $f = \mathbb{S}^\infty f^0$. In fact, we show that they are identical up to a multiplicative constant.

Theorem 3.7. *Assume $\{P_m\}_{m=1,\ldots,M-1}$ are masks for which conditions* (3.10) *hold. Then the corresponding M-scale scheme $\mathbb{S}$ is strongly convergent if and only if it is partially strongly convergent.*

Proof. If $\mathbb{S}$ is strongly convergent then by definition $\widetilde{\mathbb{S}}$ is convergent. We now assume that $\widetilde{\mathbb{S}}$ is convergent. Since conditions (3.10) hold we have for any $1 \leqslant m \leqslant M-1$ and $\gamma \in \mathbb{Z}^d$

$$\sum_{k\in\mathbb{Z}^d} p_{m,2^m k+\gamma} = C_m.$$

For $m = 2, \ldots, M-1$ we obtain from (3.9) for each dyadic point $2^{-j_0}k$

$$\lim_{j_0<j\to\infty} f^{j,m}_{2^{j-j_0}\alpha} = \lim_{j_0<j\to\infty} \sum_{r=m}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{r,2^{j-j_0}\alpha-2^r k} f_k^{j-r+1} = f_1\big(2^{-j_0}k\big) \sum_{r=m}^{M-1} C_r,$$

since for j large enough only sufficiently small neighborhoods of $2^{-j_0}k$ appear in the discrete convolutions. □

Thus, for a strongly convergent scheme we have a representation of the limit function and its components using the limit function of the partial scheme or equivalently of the first scale.

Corollary 3.8. *Let $\mathbb{S}$ be a strongly convergent M-scale subdivision scheme. Then for any initial data f^0, $\mathbb{S}^\infty f^0 = f = \sum_{m=1}^{M-1} f_m$, with*

$$f_m(x) = \left(\sum_{r=m}^{M-1} C_r\right) f_1(x), \quad m = 1, \ldots, M-1, \quad \textit{and}$$

$$f(x) = \left(\sum_{m=1}^{M-1} m C_m\right) f_1(x), \tag{3.13}$$

where $f_1 = \widetilde{\mathbb{S}}^\infty f^0$.

In the case of a strongly convergent poly-scale scheme we can use (3.13) *and choose to implement the partial scheme. If we terminate the partial algorithm at the level j, we can estimate the limit $f = \mathbb{S}^\infty f^0$ by*

$$F_k^j \approx \left(\sum_{m=1}^{M-1} mC_m\right) f_k^{j,1}.$$

In the beginning of Section 3.1 we show how a poly-scale refinable function defines a poly-scale subdivision scheme. We now show how a convergent poly-scale scheme determines a continuous poly-scale refinable function.

Theorem 3.9. *Let $\mathbb{S}$ be a quasi convergent M-scale subdivision scheme with masks $\{P_m\}$. Then it determines a compactly supported function $\phi \in C(\mathbb{R}^d)$ with the following properties:*

1. *M-scale relation*
$$\phi(x) = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,k}\phi\big(2^m x - k\big), \tag{3.14}$$
2. *Compact support*
$$\operatorname{supp}(\phi) \subseteq \left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} \operatorname{supp}(P_m)\right\rangle, \tag{3.15}$$
where $\langle X\rangle$ denotes the convex hull of the set X.
3. *PSI space of limit functions*: *for any initial data f^0 we have that*
$$\mathbb{S}^\infty f^0 = \sum_{k\in\mathbb{Z}^d} f_k^0 \phi(\cdot - k). \tag{3.16}$$

Also, if conditions (3.10) *hold then*

4. *Partition of unity*
$$\sum_{k\in\mathbb{Z}^d} \phi(\cdot - k) = 1. \tag{3.17}$$

Furthermore, if $\mathbb{S}$ is strongly convergent then ϕ is the unique solution of (3.14) *for which* (3.17) *holds.*

Proof. We select the initial data $f^0 := \{\delta_{0,k}\}$ and denote $\mathbb{S}^\infty f^0 = \phi \in C(\mathbb{R}^d)$. First we establish the compact support property (3.15). Since by (3.9) we have that $\operatorname{supp}(\phi) \subseteq \bigcup_{j\geqslant 0} 2^{-j}\langle \operatorname{supp}(f^{j,1})\rangle$ it is sufficient to consider the partial scheme.

Remark. Observe that since we only assumed that the scheme is quasi convergent, it is possible that the sequence $\{f^{j,1}\}$, $j \geqslant 0$ diverges. Yet, any bound we obtain on the supports of $f^j := f^{j,1}$ can serve as a bound for the support of ϕ.

We initialize the partial scheme with $f^{2-M} = \cdots = f^{-1} = 0$ for $M \geqslant 3$ and $f^0 := \{\delta_{0,k}\}$. From (3.8) we have that

$$f_\alpha^j = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,\alpha-2^m k} f_k^{j-m}. \tag{3.18}$$

We claim that for $j \geqslant 2 - M$

$$\operatorname{supp}\big(f^{j,1}\big) \subseteq \big(2^j - 1\big)\left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} \operatorname{supp}(P_m)\right\rangle. \tag{3.19}$$

We prove (3.19) by induction on the refinement levels. For $j = 2 - M, \dots, -1$, we have that $\operatorname{supp}(f^j) = \emptyset$. For $j = 0$, we have that $\operatorname{supp}(f^0) = \operatorname{supp}(\delta_{k,0}) = \{0\}$. Also observe that $f_k^1 = p_{1,k}$ and so (3.19) holds for all $j = 2 - M, \dots, 1$. Assume by induction that (3.19) holds for all $j' < j$ with $j > 1$. From (3.18) we can see that for each $m = 1, \dots, M-1$, the contribution to the support of f^j of the convolution $\sum_{k\in\mathbb{Z}^d} p_{m,\alpha-2^m k} f_k^{j-m}$ is contained in

$$2^m \operatorname{supp}\bigl(f^{j-m}\bigr) + \operatorname{supp}(P_m) \subseteq \bigl(2^j - 2^m\bigr)\left\langle \bigcup_{r=1}^{M-1} \frac{1}{2^r - 1} \operatorname{supp}(P_r)\right\rangle + \operatorname{supp}(P_m),$$

where the sums are Minkowski sums of sets in $\mathbb{R}^d$. Let us define the sets $X_m := (2^m - 1)^{-1} \operatorname{supp}(P_m)$, $m = 1, \dots, M-1$. Then for $1 \leqslant m \leqslant M-1$

$$\begin{aligned}\bigl(2^j - 2^m\bigr)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle + \bigl(2^m - 1\bigr)X_m &\subseteq \bigl(2^j - 2^m\bigr)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle + \bigl(2^m - 1\bigr)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle \\ &= \bigl(2^j - 1\bigr)\left\langle \bigcup_{r=1}^{M-1} X_r\right\rangle,\end{aligned}$$

which is independent of m. We can now derive (3.19) since

$$\operatorname{supp}\bigl(f^j\bigr) \subseteq \bigl(2^j - 1\bigr)\left\langle \bigcup_{m=1}^{M-1} X_m\right\rangle = \bigl(2^j - 1\bigr)\left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} \operatorname{supp}(P_m)\right\rangle.$$

Therefore, since the value of f_k^j is attached to the parameter $2^{-j}k$, we obtain

$$\operatorname{supp}(\phi) \subseteq \bigcup_{j\geqslant 0} 2^{-j}\bigl\langle \operatorname{supp}\bigl(f^j\bigr)\bigr\rangle \subseteq \left\langle \bigcup_{m=1}^{M-1} \frac{1}{2^m - 1} \operatorname{supp}(P_m)\right\rangle,$$

and so (3.15) holds.

The representation (3.16) is a consequence of the linearity of $\mathbb{S}$ and the compact support of ϕ. Next we verify (3.14). It is easy to see that after the first iteration of the full subdivision process on the initial data $f^{0,1} = \{\delta_{0,k}\}$, $f^{0,m} = 0$, $m = 2, \dots, M-1$, we have

$$f_k^{1,m} = p_{m,k}, \quad k \in \mathbb{Z}^d,\ 1 \leqslant m \leqslant M-1. \tag{3.20}$$

We now separate the scales of (3.20) and define for each m_0, $1 \leqslant m_0 \leqslant M-1$, initial control points $g_{[m_0]}^{1,m}$, $1 \leqslant m \leqslant M-1$

$$\bigl(g_{[m_0]}^{1,m}\bigr)_k := \begin{cases} p_{m_0,k}, & m = m_0, \\ 0, & 1 \leqslant m \leqslant M-1,\ m \neq m_0. \end{cases}$$

After $m_0 - 1$ iterations on this initial data, we get

$$\bigl(g_{[m_0]}^{m_0,m}\bigr)_k := \begin{cases} p_{m_0,k}, & m = 1, \\ 0, & m \neq 1. \end{cases}$$

Therefore, by dilating (3.16) we obtain that the limit is

$$\sum_{k\in\mathbb{Z}^d} p_{m_0,k}\phi\bigl(2^{m_0} \cdot - k\bigr). \tag{3.21}$$

Again, by the linearity of the scheme and the compact support of ϕ, we can sum up the limits (3.21) to obtain (3.14).

Assume that conditions (3.10) hold. To prove that ϕ has the partition of unity property (3.17), we choose $f^0 \equiv 1$. Since the subdivision algorithm is initialized by $f^{0,1} = f^0$ and $f^{0,2} = \cdots = f^{0,M-1} = 0$, the initial sum of the scales is $F^0 \equiv 1$. We show by induction that $F^j \equiv 1$, $j \geqslant 0$. Assume that after the jth iteration, each of the scales is constant, $f_k^{j,m} = \beta_{j,m}$, $m = 1, \dots, M-1$, $k \in \mathbb{Z}^d$, such that $\sum_{m=1}^{M-1} \beta_{j,m} = 1$, implying that $F^j \equiv 1$.

We now compute the sum at the $(j+1)$th level

$$\begin{aligned} F_\alpha^{j+1} &= \sum_{m=1}^{M-1} f_{2^{m-1}\alpha}^{j+1,m} \\ &= \sum_{m=1}^{M-2} f_{2^{m-1}\alpha}^{j,m} + \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,2^{m-1}\alpha-2^m k} f_k^{j,1} \\ &= \sum_{m=1}^{M-2} \beta_{j,m} + \beta_{j,1} \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,2^{m-1}\alpha-2^m k} \\ &= \sum_{m=1}^{M-1} \beta_{j,m} = 1. \end{aligned}$$

Thus,

$$\sum_{k\in\mathbb{Z}^d} \phi(\cdot - k) = \sum_{k\in\mathbb{Z}^d} f_k^0 \phi(\cdot - k) = S^\infty f^0 = \lim_{j\to\infty} F^j \equiv 1.$$

Finally, if $\mathbb{S}$ is strongly convergent then the uniqueness of ϕ as a solution of (3.14) for which (3.17) holds can be derived using the same methods as in [9]. □

Remark. In the proof of the last theorem, we used a rather complex argument to bound the support size of the function $\phi = \mathbb{S}^\infty \phi$. Recall that we have already presented the bound (3.15) using a simpler approach in Section 2 (see (2.20)). However, the technique of Section 2 uses directly the poly-scale relation (3.14) to obtain a bound in case $d = 1$, whereas here the bound holds for $d \geqslant 1$.

The above result shows the connection between poly-scale subdivision schemes and poly-scale refinability: for each quasi convergent poly-scale scheme there exists a corresponding continuous $\mathbb{S}$-*refinable function* which is a solution of the poly-scale functional equation defined by the scheme's masks. As explained in the next section this also leads to a relation between poly-scale subdivision and matrix subdivision.

We conclude this section by presenting methods to compute the $\mathbb{S}$-refinable function $\phi = \mathbb{S}^\infty \delta$. Obviously, we can use the method of proof of Theorem 3.9, initialize the subdivision algorithm with $\{\delta_{k,0}\}$ and converge to ϕ. Observe that for schemes that are only quasi convergent this approach can be unstable. Namely, while the sum of the scales converges to ϕ, the scales themselves can 'blow-up.' An alternative approach, which is known to work well for two-scale subdivision, is to first compute the values $\phi(k)$, $k \in \mathbb{Z}^d$ and then use the poly-scale relation (3.14) to recursively compute values at finer dyadic points. For example, assuming the values $\phi(k)$, $k \in \mathbb{Z}^d$ are known, the first iteration that produces the values at the half integers is

$$\begin{aligned} \phi\bigl(2^{-1}n\bigr) &= \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,k} \phi\bigl(2^m 2^{-1} n - k\bigr) \\ &= \sum_{m=0}^{M-2} \sum_{k\in\mathbb{Z}^d} p_{m+1,k} \phi\bigl(2^m n - k\bigr), \quad n \in \mathbb{Z}^d. \end{aligned} \tag{3.22}$$

Since ϕ has compact support, the sum in (3.22) is well defined as only a finite number of values $\{\phi(2^m n - k)\}$ are non-zero. Thus, we only need to describe how the values at the points $k \in \mathbb{Z}^d$ can be computed directly from the masks $\{P_m\}_{m=1}^{M-1}$. We first observe that the case of $\phi(k) = 0$ for all $k \in \mathbb{Z}^d$ is not possible. Otherwise by the above recursive method we have that $\phi(2^{-j}k) = 0$ for any dyadic point $2^{-j}k$ with $j \geqslant 0$, $k \in \mathbb{Z}^d$, which by continuity leads to $\phi(x) = 0$ for all $x \in \mathbb{R}^d$, a contradiction. Thus, with $\Lambda := \{\alpha \mid \alpha \in \operatorname{supp}(\phi) \cap \mathbb{Z}^d\}$, the set $\{\phi(\alpha) \mid \alpha \in \Lambda\}$ is not empty. By assuming some order on Λ, we can define the vector $V_\phi \in \mathbb{R}^{|\Lambda|}$, $V_\phi := (\phi(\alpha))_{\alpha\in\Lambda}$. Also, for each $\alpha \in \Lambda$ we have that,

$$\phi(\alpha) = \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}^d} p_{m,k}\phi\bigl(2^m\alpha - k\bigr) = \sum_{m=1}^{M-1} \sum_{\beta\in\Lambda} p_{m,2^m\alpha-\beta}\phi(\beta)$$
$$= \sum_{\beta\in\Lambda} \phi(\beta) \sum_{m=1}^{M-1} p_{m,2^m\alpha-\beta}.$$

Therefore, the values of the $\mathbb{S}$-refinable function at the integers correspond to a left eigenvector of the matrix

$$A := \left(\sum_{m=1}^{M-1} p_{m,2^m\alpha-\beta}\right)_{\alpha,\beta\in\Lambda},$$

with eigenvalue 1. By Theorem 3.9 if $\mathbb{S}$ is a strongly convergent scheme the $\mathbb{S}$-refinable function is unique, which implies that the eigenvector subspace corresponding to the eigenvalue 1 is of dimension 1.

Following [5], this approach can also be used to compute derivatives of the $\mathbb{S}$-refinable function. If ϕ is known to be in $C^m(\mathbb{R}^d)$ with $m \geqslant 0$, then for each homogeneous differential operator D^γ with $|\gamma| \leqslant m$ the vector $V_{\phi,\gamma} := ((D^\gamma\phi)(\alpha))_{\alpha\in\Lambda}$ is a left eigenvector of the matrix

$$A_{D^\gamma} := \left(\sum_{m=1}^{M-1} 2^{m|\gamma|} p_{m,2^m\alpha-\beta}\right)_{\alpha,\beta\in\Lambda},$$

corresponding to eigenvalue 1.

3.2. Poly-scale subdivision and matrix subdivision

We are now ready to see how poly-scale subdivision can be represented as two-scale matrix subdivision. We follow [4] for basic results on matrix subdivision. Assume that $\mathbb{S}$ is a quasi convergent M-scale scheme given by masks $\{P_m\}_{m=1,\dots,M-1}$. By Theorem 3.9 the scheme has an $\mathbb{S}$-refinable function $\phi \in C(\mathbb{R}^d)$ with an M-scale relation (3.14). We now go back to the construction (2.17) and define

$$\Sigma = \bigcup_{m=0}^{M-2} \Phi_m, \quad \Phi_m := \bigl\{\phi\bigl(2^m\cdot - r\bigr) \bigm| r \in \bigl\{0,\dots,2^m-1\bigr\}^d\bigr\}. \tag{3.23}$$

As shown in the previous chapter, the FSI space $S(\Sigma)$ is two-scale refinable which, assuming some order on Σ, implies the existence of a two-scale relation

$$\Sigma = \sum_{k\in\mathbb{Z}^d} A_k \Sigma(2\cdot - k)^t, \quad A_k \in M_{|\Sigma|\times|\Sigma|}(\mathbb{R}). \tag{3.24}$$

The corresponding $|\Sigma|$th dimensional matrix subdivision process is defined as follows. We use the matrices $\{A_k\}$ of (3.24) as the masks of the subdivision algorithm. For any given initial sequence of data vectors $\vec{f}_k^{\,0} \in \mathbb{R}^{|\Sigma|}$, $k \in \mathbb{Z}^d$, we iterate

$$\vec{f}_\alpha^{\,j} = \sum_{k\in\mathbb{Z}^d} \vec{f}_k^{\,j-1} A_{\alpha-2k}.$$

It can be shown that if $\mathbb{S}$ is strongly convergent then the matrix subdivision scheme also converges. In such a case the limit of the matrix subdivision scheme for initial vector data $\vec{f}^{\,0} = \{\vec{f}_k^{\,0}\}_{k\in\mathbb{Z}^d}$ is

$$\vec{f}(x) = \sum_{k\in\mathbb{Z}^d} \vec{f}_k^{\,0} \Sigma(x-k).$$

Example 3.10. Let $\mathbb{S}$ be a univariate three-scale quasi convergent scheme with an $\mathbb{S}$-refinable function $\phi \in C(\mathbb{R})$. Let $P_1 = \{p_{1,k}\}_{k\in\mathbb{Z}}$, $P_2 = \{p_{2,k}\}_{k\in\mathbb{Z}}$ be the two masks of $\mathbb{S}$. In this case $\Sigma = (\phi, \phi(2\cdot), \phi(2\cdot -1))$ and we have the following two-scale relation

$$\big(\phi, \phi(2\cdot), \phi(2\cdot -1)\big)^t = \sum_{k\in\mathbb{Z}} A_k\big(\phi(2\cdot -k), \phi(4\cdot -2k), \phi\big(4\cdot -(2k+1)\big)\big)^t,$$

where

$$A_k = \begin{cases} \begin{bmatrix} p_{1,0} & p_{2,0} & p_{2,1} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & k = 0, \\ \begin{bmatrix} p_{1,1} & p_{2,2} & p_{2,3} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, & k = 1, \\ \begin{bmatrix} p_{1,k} & p_{2,2k} & p_{2,2k+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \text{else}. \end{cases} \tag{3.25}$$

As one can see, the structure of poly-scale subdivision is somewhat hidden in the degenerate structure of the mask of the two-scale matrix subdivision. To strengthen this observation let us see the following example where we show the equivalence of the necessary conditions for uniform convergence.

Example 3.11. Let $\mathbb{S}$ be a univariate three-scale scheme with masks $P_1 = \{p_{1,k}\}_{k\in\mathbb{Z}}$, $P_2 = \{p_{2,k}\}_{k\in\mathbb{Z}}$. First assume that the masks of $\mathbb{S}$ satisfy the necessary conditions (3.10). As we have seen, the corresponding matrix subdivision, denoted by $\mathbb{S}_M$, is defined by the matrices $\{A_k\}$ of the form (3.25). By [4, Proposition 2.2] a necessary condition for $\mathbb{S}_M$ to converge uniformly is that the matrices

$$B_0 := \sum_{k\in\mathbb{Z}} A_{2k}^t, \qquad B_1 := \sum_{k\in\mathbb{Z}} A_{2k+1}^t,$$

have a joint eigenvector corresponding to the eigenvalue 1. For this special type of matrix subdivision the matrices B_0, B_1 can be easily computed using (3.10) and (3.25)

$$B_0 = \sum_{k\in\mathbb{Z}} A_{2k}^t = \begin{pmatrix} \sum_{k\in\mathbb{Z}} p_{1,2k} & 1 & 0 \\ \sum_{k\in\mathbb{Z}} p_{2,4k} & 0 & 0 \\ \sum_{k\in\mathbb{Z}} p_{2,4k+1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 1 & 0 \\ C_2 & 0 & 0 \\ C_2 & 0 & 0 \end{pmatrix},$$
$$B_1 = \sum_{k\in\mathbb{Z}} A_{2k+1}^t = \begin{pmatrix} \sum_{k\in\mathbb{Z}} p_{1,2k+1} & 0 & 1 \\ \sum_{k\in\mathbb{Z}} p_{2,4k+2} & 0 & 0 \\ \sum_{k\in\mathbb{Z}} p_{2,4k+3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 & 1 \\ C_2 & 0 & 0 \\ C_2 & 0 & 0 \end{pmatrix}.$$

Since $C_1 + C_2 = 1$, it is easy to see that $(1, C_2, C_2)$ is a joint eigenvector for B_0, B_1, corresponding to the eigenvalue 1. The opposite is also true. Assume that $x = (x_1, x_2, x_3)$ is a joint eigenvector for B_0, B_1 corresponding to the eigenvalue 1. Since B_0, B_1 have the form

$$B_0 = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} \beta_1 & 0 & 1 \\ \beta_2 & 0 & 0 \\ \beta_3 & 0 & 0 \end{pmatrix},$$

we can apply B_0, B_1 to x and verify that $\alpha_1 = \beta_1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3$, and $\alpha_1 + \alpha_2 = 1$. The last conditions are exactly the necessary conditions for a strongly convergent three-scale scheme.

In the analysis of matrix subdivision the notion of stability is important [4,13,14]. However, it turns out that the set Σ of (3.23) is not an L_p-stable basis for $S(\Sigma)$ in the case of poly-scale subdivision with more than two scales.

Definition 3.12. Let $S(\Phi)$ be an FSI space in $L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$. The set Φ is called an *L_p-stable generating set* (for its span) if there exist constants $0 < A \leqslant B < \infty$ such that for every $c \in l_p(\Phi \times \mathbb{Z}^d)$

$$A\|c\|_{l_p(\Phi\times\mathbb{Z}^d)} \leqslant \Bigl\| \sum_{\phi\in\Phi,\ k\in\mathbb{Z}^d} c_{\phi,k}\phi(\cdot - k) \Bigr\|_{L_p(\mathbb{R}^d)} \leqslant B\|c\|_{l_p(\Phi\times\mathbb{Z}^d)}. \tag{3.26}$$

Theorem 3.13. *Assume $\mathbb{S}$ is a quasi convergent M-scale scheme, $M > 2$, for which conditions* (3.10) *hold. Let $\phi \in C(\mathbb{R}^d)$ be the corresponding $\mathbb{S}$-refinable function. Then the set Σ of* (3.23) *is not an L_∞-stable basis for $S(\Sigma)$.*

Proof. Assume that the set Σ is L_∞-stable. This means that there exists a constant $A > 0$ such that for any $\{g_{m,k}\colon k \in \mathbb{Z}^d\} \in l_\infty(\mathbb{Z}^d)$, $m = 0, \ldots, M-2$ we have that

$$A \max_{0\leqslant m\leqslant M-2} \bigl\|\{g_{m,k}\}\bigr\|_{l_\infty(\mathbb{Z}^d)} \leqslant \Bigl\| \sum_{m=0}^{M-2} \sum_{k\in\mathbb{Z}^d} g_{m,k}\phi\bigl(2^m x - k\bigr) \Bigr\|_{L_\infty(\mathbb{R}^d)}. \tag{3.27}$$

Let $B_j := \{\beta_{j,m}\}_{m=0}^{M-2}$, $j \geqslant 1$, be a sequence of vectors in $\mathbb{R}^{M-1}$ such that $\beta_{j,0} \underset{j\to\infty}{\to} +\infty$ and $\sum_{m=0}^{M-2} \beta_{j,m} = 1$. The partition of unity and the compact support properties of ϕ imply that

$$\sum_{m=0}^{M-2} \beta_{j,m} \sum_{k\in\mathbb{Z}^d} \phi\bigl(2^m \cdot - k\bigr) = 1.$$

This leads to a contradiction since by (3.27) with $g_{m,k}^{(j)} = \beta_{j,m}$, $k \in \mathbb{Z}^d$,

$$1 = \Bigl\| \sum_{m=0}^{M-2} \beta_{j,m} \sum_{k\in\mathbb{Z}^d} \phi\bigl(2^m \cdot - k\bigr) \Bigr\|_{L_\infty(\mathbb{R}^d)} \geqslant A\beta_{j,0} \underset{j\to\infty}{\to} +\infty.$$

Therefore the set Σ is not L_∞-stable. □

Remark. Using a similar approach, it is easy to see that Σ is also not L_p-stable for all $1 \leqslant p \leqslant \infty$.

3.3. Analysis of convergence and smoothness, the univariate case

In this section we assume the dimension $d = 1$. In our analysis we frequently make use of the z-transform. For any given data $f = \{f_k\}_{k\in\mathbb{Z}}$ and mask $P_m := \{p_{m,k}\}_{k\in\mathbb{Z}}$, the convolution $\sum_{k\in\mathbb{Z}} p_{m,\alpha-2^m k} f_k$, $\alpha \in \mathbb{Z}$, can be represented in z-transform notation by $P_m(z) f(z^{2^m})$ where $P_m(z) = \sum_{k\in\mathbb{Z}} p_{m,k} z^k$ and $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$. With the z-transform the $(j+1)$th iteration of the poly-scale subdivision (3.4) can be represented by

$$f^{j+1,m}(z) = \begin{cases} f^{j,m+1}(z) + P_m(z) f^{j,1}\bigl(z^{2^m}\bigr), & 1 \leqslant m \leqslant M-2, \\ P_{M-1}(z) f^{j,1}\bigl(z^{2^{M-1}}\bigr), & m = M-1. \end{cases} \tag{3.28}$$

Lemma 3.14. *If the masks $\{P_m\}_{m=1}^{M-1}$ satisfy conditions* (3.10)*, then for $m = 1, \ldots, M-1$ the polynomial $P_m(z)$ is divisible by $\prod_{n=0}^{m-1}(1 + z^{2^n})$.*

Proof. Fix m, $1 \leqslant m \leqslant M-1$. By virtue of (3.10), we have

$$\sum_{k\in\mathbb{Z}} P_{m,2^m k+\gamma} = C_m, \quad \gamma \in E_m^1 = \bigl\{0, 1, \ldots, 2^m - 1\bigr\}.$$

Since the mask P_m is of finite support, $P_m(z)$ is a Laurent polynomial. Thus, we can change the order of summation and rewrite $P_m(z)$ as

$$P_m(z) = \sum_{\gamma=0}^{2^m-1} \sum_{k\in\mathbb{Z}} p_{m,2^m k+\gamma} z^{2^m k+\gamma} = \sum_{\gamma=0}^{2^m-1} z^\gamma \sum_{k\in\mathbb{Z}} p_{m,2^m k+\gamma} z^{2^m k}.$$

Assume $\xi \neq 1$ with $\xi^{2^m} = 1$. Then,

$$P_m(\xi) = C_m \sum_{\gamma=0}^{2^m-1} \xi^\gamma = C_m \frac{1-\xi^{2^m}}{1-\xi} = 0.$$

Since $P_m(z) = 0$ for any $\xi \neq 1$ which is a 2^mth root of unity, we have that $(1-z^{2^m})/(1-z)$ divides $P_m(z)$. It is easy to see that

$$\frac{1-z^{2^m}}{1-z} = \prod_{n=0}^{m-1} \bigl(1+z^{2^n}\bigr). \qquad \square$$

Theorem 3.15. *Let $\mathbb{S}$ be an M-scale subdivision scheme with masks $\{P_m\}$ satisfying conditions* (3.10)*. Then there exists an M-scale subdivision scheme $D^{[1]}\mathbb{S}$ which generates the first order divided differences*

$$\bigl(df^{j+1,1}, \ldots, df^{j,M-1}\bigr) = D^{[1]}\mathbb{S}\bigl(df^{j,1}, \ldots, df^{j,M-1}\bigr),$$

where

$$\bigl(df^{j,m}\bigr)_k := 2^{j+m-1}\Delta f_k^{j,m}, \quad \Delta f_k^{j,m} := \bigl(f_{k+1}^{j,m} - f_k^{j,m}\bigr).$$

The masks of this scheme are given by

$$Q_m(z) := \frac{2^m z^{2^m-1}}{\prod_{n=0}^{m-1}(1+z^{2^n})} P_m(z), \quad m = 1, \ldots, M-1. \tag{3.29}$$

Proof. We need only prove for $M \geqslant 3$, since the case $M = 2$ is treated in [9, Proposition 3.1]. Let $f^{j,m}(z) = \sum_{k\in\mathbb{Z}} f_k^{j,m} z^k$. Then the z-transform of the divided difference sequence $h^{j,m} := df^{j,m}$ is of the form

$$h^{j,m}(z) := 2^{j+m-1}\frac{1-z}{z} f^{j,m}(z), \quad m = 1, \ldots, M-1.$$

Therefore,

$$f^{j,m}(z) = 2^{-(j+m-1)}\frac{z}{1-z} h^{j,m}(z).$$

Since $\{P_m\}$ satisfy (3.10), by Lemma 3.14 the masks defined in (3.29) are Laurent polynomials. As we assumed $M \geqslant 3$ there are two cases. For $1 \leqslant m \leqslant M-2$ we obtain from (3.28)

$$\begin{aligned} h^{j+1,m}(z) &= 2^{j+m}\frac{1-z}{z} f^{j+1,m}(z) \\ &= 2^{j+m}\frac{1-z}{z}\bigl(f^{j,m+1}(z) + P_m(z) f^{j,1}\bigl(z^{2^m}\bigr)\bigr) \\ &= 2^{j+m}\frac{1-z}{z}\left(2^{-(j+m)}\frac{z}{1-z}h^{j,m+1}(z) + P_m(z)2^{-j}\frac{z^{2^m}}{1-z^{2^m}}h^{j,1}\bigl(z^{2^m}\bigr)\right) \\ &= h^{j,m+1}(z) + Q_m(z)h^{j,1}. \end{aligned}$$

Similarly, for $m = M-1$ we obtain from (3.28)

$$h^{j+1,M-1}(z) = Q_{M-1}(z)h^{j,1}\bigl(z^{2^{M-1}}\bigr).$$

We conclude that the scheme defined by the masks $\{Q_m\}_{m=1}^{M-1}$ is the required $D^{[1]}\mathbb{S}$. $\square$

Definition 3.16. Let $\mathbb{S}$ be an M-scale subdivision scheme with masks satisfying conditions (3.10) and let $D^{[1]}\mathbb{S}$ be the corresponding *divided difference scheme*, defined by the masks $\{Q_m\}$ in (3.29). The M-scale *difference scheme* $\Delta\mathbb{S}$ is defined by $\{2^{-m}Q_m\}$. It generates the differences $\Delta f^{j,m} := (f_{k+1}^{j,m} - f_k^{j,m})$.

In the two-scale subdivision case a well established technique for determining the convergence of a scheme is analyzing the corresponding difference scheme. Here we generalize this approach to the poly-scale case. First, we require the following three simple results. The proofs can be found in Appendix A.

Lemma 3.17. *Let $\xi \neq 1$ be a 2^mth root of unity for some $m \geqslant 1$. Then, $\xi^{2^n} = -1$ for some $0 \leqslant n < m$.*

Lemma 3.18. *Let $C_m > 0$, $m = 1, \ldots, M-1$, satisfy $\sum_{m=1}^{M-1} C_m = 1$. Let $\{a_n\}_{n \geqslant 1}$ be a sequence of real numbers such that for $n \geqslant M$*

$$\left| a_n - \sum_{m=1}^{M-1} C_m a_{n-m} \right| \leqslant C\mu^n,$$

for some $0 < \mu < 1$. Then, $|a_n - a_{n-1}| \leqslant \widetilde{C}\tilde{\mu}^n$, where $a\tilde{\mu} < 1$ and $\tilde{c}$, $\tilde{\mu}$ do not depend on the sequence $\{a_n\}$.

For $M = 3$ the positivity condition in Lemma 3.18 can be relaxed.

Lemma 3.19. *Let $p, q \in \mathbb{R}$ with $0 < p < 2$, $p + q = 1$ and let $\{a_n\}_{n \geqslant 1}$ be a sequence of real numbers. If $|a_n - (pa_{n-1} + qa_{n-2})| \leqslant C\mu^n$ for some $0 < \mu < 1$ then $|a_n - a_{n-1}| \leqslant \widetilde{C}\tilde{\mu}^n$, where $a\tilde{\mu} < 1$ and $\tilde{c}$, $\tilde{\mu}$ do not depend on the sequence $\{a_n\}$.*

Theorem 3.20. *Let $\mathbb{S}$ be a poly-scale scheme with masks satisfying* (3.10)*, Then,*

1. *If $\mathbb{S}$ is strongly convergent then $\Delta\mathbb{S}$ converges strongly to zero for any initial data.*
2. *If $\Delta\mathbb{S}$ converges strongly to zero for any initial data then*
   a. *$\mathbb{S}$ is quasi convergent.*
   b. *If $C_m > 0$, for each m, $m = 1, \ldots, M-1$, where $\{C_m\}$ are defined in* (3.10)*, then $\mathbb{S}$ is strongly convergent.*
   c. *If $\mathbb{S}$ is a three-scale scheme and $0 < C_1 < 2$ then $\mathbb{S}$ is strongly convergent.*

Proof. To make the proof shorter we assume that $M \geqslant 3$. The proof of the case $M = 2$ is given in [9, Proposition 3.2]. The proof of the first direction simply requires the use of the triangle inequality. To see the (weaker) opposite direction 2.a we begin by observing that if $\Delta\mathbb{S}$ converges strongly to zero on some initial input, then the partial difference scheme $\widetilde{\Delta\mathbb{S}}$ generating only $\Delta f^{j,1}$ also converges uniformly to zero on the same input.

Remark. It is easy to see that the converse is also true. Namely, the convergence to zero of $\widetilde{\Delta\mathbb{S}}$ implies the strong convergence to zero of $\Delta\mathbb{S}$. This is because by (3.9)

$$\Delta f^{j,m}(z) = \sum_{r=m}^{M-1} 2^{-r} Q_r(z) \Delta f^{j-r+1,1}\big(z^{2^r}\big), \quad m = 2, \ldots, M-1,$$

where $\{Q_m\}$ are defined by (3.29).

It can be shown that the uniform convergence of $\widetilde{\Delta\mathbb{S}}$ to zero implies that there exists $0 < \mu < 1$ and a power J_0 such that for all initial data $f^0 \in l_\infty(\mathbb{Z})$

$$\big\| \widetilde{\Delta\mathbb{S}}^{J_0} f^0 \big\|_\infty < \mu \big\| f^0 \big\|_\infty. \tag{3.30}$$

We now use (3.5) to prove uniform quasi convergence. Denoting

$$P_m *_m f := \left(\sum_{\alpha \in \mathbb{Z}} p_{m,k-2^m\alpha} f_\alpha \right)_k$$

we consider

$$\begin{aligned}
F^{j+1}(x) - F^{j}(x) &= \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}} f_k^{j+1,m} H\bigl(2^{j+m}x - k\bigr) \\
&\quad - \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}} f_k^{j,m} H\bigl(2^{j+m-1}x - k\bigr) \\
&= \sum_{m=1}^{M-2} \sum_{k\in\mathbb{Z}} \bigl(f_k^{j,m+1} + \bigl(P_m *_m f^{j,1}\bigr)_k\bigr) H\bigl(2^{j+m}x - k\bigr) \\
&\quad + \sum_{k\in\mathbb{Z}} \bigl(P_{M-1} *_{M-1} f^{j,1}\bigr)_k H\bigl(2^{j+M-1}x - k\bigr) \\
&\quad - \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}} f_k^{j,m} H\bigl(2^{j+m-1}x - k\bigr) \\
&= \sum_{m=1}^{M-1} \sum_{k\in\mathbb{Z}} \bigl(P_m *_m f^{j,1}\bigr)_k H\bigl(2^{j+m}x - k\bigr) \\
&\quad - \sum_{k\in\mathbb{Z}} f_k^{j,1} H\bigl(2^{j}x - k\bigr).
\end{aligned}$$

Denote by $U(z) = z^{-1}/2 + 1 + z/2$ the mask of H and let $U^N(z) := \prod_{n=1}^{N} U(z^{2^{n-1}})$. Then, since

$$\sum_{k\in\mathbb{Z}} a_k H\bigl(2^l x - k\bigr) = \sum_{k\in\mathbb{Z}} b_k H\bigl(2^{l+N}x - k\bigr),$$

with $B(z) = U^N(z)A(z^{2^n})$, where $A(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ and $B(z) = \sum_{k\in\mathbb{Z}} b_k z^k$, with a slight abuse of the convolution notation we can write

$$\begin{aligned}
F^{j+1}(x) - F^{j}(x) &= \sum_{k\in\mathbb{Z}} \left(\left(\sum_{m=1}^{M-1} U^{M-m-1} * P_m - U^{M-1} \right) * f^{j,1} \right)_k \\
&\quad \times H\bigl(2^{j+M-1}x - k\bigr).
\end{aligned}$$

Therefore, with

$$D(z) := \prod_{m=1}^{M-1} \left(\sum_{r=1}^{M-m-1} U\bigl(z^{2^{r-1}}\bigr) \right) P_m\bigl(z^{2^{M-m-1}}\bigr) - \prod_{m=1}^{M-1} U\bigl(z^{2^{m-1}}\bigr), \tag{3.31}$$

the nodes of the polygonal line which is the graph of $F^{j+1}(x) - F^{j}(x)$ are the coefficients of $D(z) f^{j,1}(z^{2^{M-1}})$.

Next we show that $D(z)$ vanishes at all the 2^{M-1}th roots of unity. Observe that the mask U (as a two-scale scheme) and the masks $\{P_m\}_{m=1}^{M-1}$ fulfill the necessary conditions (3.10). In particular, this implies that $U(1) = 2$, $U(-1) = 0$. We begin with the case $\xi = 1$. Using (3.31) we have,

$$D(1) = \sum_{m=1}^{M-1} 2^{M-m-1} 2^m C_m - 2^{M-1} = 2^{M-1} \sum_{m=1}^{M-1} C_m - 2^{M-1} = 0.$$

Now assume that $\xi \neq 1$ is a 2^{M-1}th root of unity. First we analyze the product $\prod_{m=1}^{M-1} U(\xi^{2^{m-1}})$ appearing in (3.31). By Lemma 3.17 for some $1 \leqslant m \leqslant M-1$ we must have $\xi^{2^{m-1}} = -1$. Since $U(-1) = 0$, this product is zero for any choice of such a root $\neq 1$. We now prove that the sum appearing in (3.31) is also zero. We analyze separately each term $(\prod_{r=1}^{M-m-1} U(\xi^{2^{r-1}})) P_m(\xi^{2^{M-m-1}})$ for $1 \leqslant m \leqslant M-1$. Observe that since ξ is a 2^{M-1}th root of unity, $\xi^{2^{M-m-1}}$ is a 2^mth root of unity. There are two cases: if $\xi^{2^{M-m-1}} \neq 1$, then by Lemma 3.14 $P_m(\xi^{2^{M-m-1}}) = 0$. Else $\xi^{2^{M-m-1}} = 1$. In such a case, using again Lemma 3.17, we must have $(\prod_{r=1}^{M-m-1} U(\xi^{2^{r-1}})) = 0$. Combining the last two arguments we conclude that $D(\xi)=0$ for $\xi \neq 1$ that is a 2^{M-1}th root of unity. Thus, $D(z)$ can be factorized as follows:

$$D(z)=\frac{1-z^{2^{M-1}}}{z^{2^{M-1}}}E(z),$$

where $E(z)=\sum_{k\in\mathbb{Z}} e_k z^k$ is a polynomial. We see that

$$\begin{aligned} D(z)f^{j,1}\big(z^{2^{M-1}}\big) &= E(z)\frac{1-z^{2^{M-1}}}{z^{2^{M-1}}} f^{j,1}\big(z^{2^{M-1}}\big) \\ &= E(z)\sum_{k\in\mathbb{Z}}\big(f^{j,1}_{k+1}-f^{j,1}_k\big)z^{2^{M-1}k}. \end{aligned} \tag{3.32}$$

We can now conclude from the convergence to zero of $\widetilde{\Delta\mathbb{S}}$ that $\{F^j(x)\}$ is a Cauchy sequence. Using (3.30) and (3.32) and denoting

$$C_E := \max_{\gamma\in E^1_{M-1}}\left\{\sum_{k\in\mathbb{Z}}|e_{\gamma-2^{M-1}k}|\right\},$$

we get

$$\begin{aligned} \big\|F^{j+1}(x)-F^j(x)\big\|_\infty &\leqslant \big\|D(z)f^{j,1}\big(z^{2^{M-1}}\big)\big\|_\infty \\ &\leqslant C_E\big\|\widetilde{\Delta\mathbb{S}}^j f^0\big\|_\infty \\ &\leqslant C_E \max_{0\leqslant r\leqslant J_0}\big\|\widetilde{\Delta\mathbb{S}}^r f^0\big\|_\infty \mu^{[j/J_0]} \\ &\leqslant C\big(\mathbb{S},f^0\big)\eta^j, \end{aligned}$$

with $\eta:=\mu^{1/J_0}<1$. Since $\{F^j(x)\}$ are continuous and converge uniformly, the limit $f(x)$ is also continuous. Consequently, $\mathbb{S}$ is quasi convergent.

We now prove claims 2.b and 2.c. Let us denote

$$f^j(x):=\sum_{k\in\mathbb{Z}^d} f^{j,1}_k H\big(2^j x-k\big).$$

We claim that if $\Delta\mathbb{S}$ converges strongly to zero for any initial data, then $f^j(x)-\sum_{m=1}^{M-1}C_m f^{j-m}(x)$ is a Cauchy sequence. Indeed, using the same technique used to prove 2.a one can show that there exist constants $B>0$ and η, $0<\eta<1$ such that for all $x\in\mathbb{R}$

$$\left|f^j(x)-\sum_{m=1}^{M-1}C_m f^{j-m}(x)\right|\leqslant B\eta^j, \quad j\geqslant 2.$$

To prove 2.b we apply Lemma 3.18. To prove 2.c we apply Lemma 3.19. □

Example 3.21. To see that the convergence to zero of the difference scheme $\Delta\mathbb{S}$ does not imply the strong convergence of $\mathbb{S}$ but the weaker quasi convergence, we give the following example. Let $\beta\in\mathbb{R}$ and let $\mathbb{S}$ be the three-scale scheme defined by the centered masks

$$P_1=\beta\left\{\tfrac12,1,\tfrac12\right\},\qquad P_2=(1-\beta)\left\{\tfrac14,\tfrac12,\tfrac34,1,\tfrac34,\tfrac12,\tfrac14\right\}.$$

Observe that $\mathbb{S}$ is interpolatory and that $P_1(z)=\beta U(z)$, $P_2(z)=(1-\beta)U(z)U(z^2)$ where $U(z)$ is the mask corresponding to the hat function H. As we shall now see, this means that $\mathbb{S}$ is actually a three-scale representation of a two-scale scheme. We argue that for the initial data $f^0=\delta$, $f^1=0$ the following holds:

$$\begin{aligned} f^{j,1}(x) &:= \sum_{k\in\mathbb{Z}} f^{j,1}_k H\big(2^j x-k\big)=K_j(\beta)H(x), \\ f^{j,2}(x) &:= \sum_{k\in\mathbb{Z}} f^{j,2}_k H\big(2^{j+1}x-k\big)=\big(1-K_j(\beta)\big)H(x), \end{aligned} \tag{3.33}$$

with

$$K_j(\beta) := \frac{(\beta-1)^{j+1}-1}{\beta-2}, \quad j \geqslant 0.$$

If (3.33) holds then we have that $F^j(x) = f^{j,1}(x) + f^{j,2}(x) = H$ for $j \geqslant 0$, which implies that $\mathbb{S}$ is quasi convergent. We verify (3.33) using induction. It is easy to see that for $j = 0$, with the initial data $f^{0,1} = \delta$, $f^{0,2} = 0$, (3.33) holds since $K_0(\alpha) = 1$. Assume that (3.33) holds for j. We have

$$\begin{aligned} f^{j+1,1}(x) &= f^{j,2}(x) + \sum_{k\in\mathbb{Z}}\biggl(\sum_{\alpha\in\mathbb{Z}} p_{1,k-2\alpha} f_\alpha^{j,1}\biggr) H\bigl(2^{j+1}x-k\bigr) \\ &= f^{j,2}(x) + \beta f^{j,1}(x) \\ &= \bigl(1-K_j(\beta)\bigr)H(x) + \beta K_j(\beta)H(x) \\ &= K_{j+1}(\beta)H(x). \end{aligned}$$

Also

$$\begin{aligned} f^{j+1,2}(x) &= \sum_{k\in\mathbb{Z}}\biggl(\sum_{\alpha\in\mathbb{Z}} p_{2,k-4\alpha} f_\alpha^{j,1}\biggr) H\bigl(2^{j+1}x-k\bigr) \\ &= (1-\beta) f^{j,1}(x) \\ &= (1-\beta)K_j(\beta)H(x) \\ &= \bigl(1-K_{j+1}(\beta)\bigr)H(x). \end{aligned}$$

Thus, (3.33) holds and $\mathbb{S}$ is quasi convergent. It is easy to see that for the choice $0 < \beta < 2$ the scheme is strongly convergent as well, which is exactly what is asserted in Theorem 3.20, 2.c. Let us now choose $2 < \beta < 3$. For such a choice it is easy to see using (3.33) that $f^{j,1}(x)$, $f^{j,2}(x)$ diverge and so the scheme is not strongly convergent. Nevertheless, for $2 < \beta < 3$ the difference scheme $\Delta\mathbb{S}$ does converge strongly to zero. This is because by (3.33)

$$\begin{aligned} &\bigl|\Delta f_k^{j,1}\bigr| \leqslant 2^{-j}K_j(\beta) \leqslant C\biggl(\frac{\beta-1}{2}\biggr)^j \underset{j\to\infty}{\to} 0, \\ &\bigl|\Delta f_k^{j,2}\bigr| \leqslant 2^{-(j+1)}\bigl|1-K_j(\beta)\bigr| \underset{j\to\infty}{\to} 0. \end{aligned}$$

The following result is a generalization of [9, Theorem 3.4]. It generalizes a well known result in the two-scale subdivision: adding B-spline factors to the mask of a convergent scheme produces a convergent scheme with higher smoothness.

Theorem 3.22. *Let $\mathbb{S}$ be a univariate M-scale subdivision scheme with masks of the form*

$$P_m(z) = \biggl(\frac{\prod_{n=0}^{m-1}(1+z^{2^n})}{2^m z^{2^m-1}}\biggr)^r Q_m(z), \quad m = 1,\ldots,M-1, \tag{3.34}$$

where $r \in \mathbb{N}$. If the poly-scale scheme defined by $\{Q_m\}$ is (strongly convergent) quasi convergent, then the scheme $\mathbb{S}$ is (strongly convergent) quasi convergent. Furthermore, the $\mathbb{S}$-refinable function $\phi := \mathbb{S}^\infty\delta$ is in $C^r(\mathbb{R})$ and provides approximation order r.

Proof. Since the proofs of the strong and quasi cases are identical, we show only the case of quasi convergence. We start with $r = 1$. By Theorem 3.15, the scheme corresponding to $\{Q_m\}$ is exactly the divided difference scheme $D^{[1]}\mathbb{S}$ corresponding to $\mathbb{S}$. Recall that $D^{[1]}\mathbb{S}$ generates for any initial data f^0 the divided difference data $df^{j,m} := \{2^{j+m-1}\Delta f_k^{j,m}\}_{k\in\mathbb{Z}}$, $j \geqslant 0$, $m = 1,\ldots,M-1$. We now construct for $f^0 = \delta$ the sequence of functions

$$g_j(x) := \sum_{m=1}^{M-1} 2^{j+m-1}\sum_{k\in\mathbb{Z}}\Delta f_k^{j,m}\mathbf{1}_{[2^{-(j+m-1)}k,\,2^{-(j+m-1)}(k+1))}(x), \tag{3.35}$$

where $\mathbf{1}_{[a,b]}$ is the characteristic function of the interval $[a,b]$. It can be shown that the quasi convergence of $D^{[1]}\mathbb{S}$ implies that the sequence g_j converges to $g := (D^{[1]}\mathbb{S})^{\infty}(\Delta\delta) \in C(\mathbb{R})$. Since there exists a bounded domain Ω such that $\operatorname{supp}(g)$, $\operatorname{supp}(g_j) \subseteq \Omega$, $j \geqslant 0$ we have

$$\int_{-\infty}^{x} g_j(t)\,\mathrm{d}t \underset{j\to\infty}{\to} \int_{-\infty}^{x} g(t)\,\mathrm{d}t.$$

Let us denote $\phi(x) := \int_{-\infty}^{x} g(t)\,\mathrm{d}t$. By definition of $g_j(x)$

$$\int_{-\infty}^{x} g_j(t)\,\mathrm{d}t = \sum_{m=1}^{M-1}\sum_{k\in\mathbb{Z}} f_k^{j,m} H\bigl(2^{j+m-1}x - k\bigr) = F^j(x) \underset{j\to\infty}{\to} \phi(x),$$

which implies that $\mathbb{S}$ is quasi convergent. Since $g \in C^0(\mathbb{R})$, we also obtain that $\mathbb{S}^{\infty}\delta = \phi \in C^1(\mathbb{R})$. For $r > 1$, the quasi convergence is proved by repeated application of the above argument. The smoothness and approximation order properties follow from Corollary 2.11 and Theorem 2.12. □

Finally, we present an application of the analysis tools of this section to the second part of Examples 3.6.

Example 3.23. Let $\{\mathbb{S}_\beta\}$ be the parametric family defined by the masks

$$P_{\beta,1}(z) = \frac{\beta}{2} z^{-1}(1+z)^2, \qquad P_{\beta,2}(z) = \frac{1-\beta}{16} z^{-4}(1+z)^4\bigl(1+z^2\bigr)^2.$$

Then, for the range $0 \leqslant \beta < 1$, the schemes is strongly convergent. For the range $-1/3 < \beta < 0$ the scheme is quasi convergent. For the range $-1/3 < \beta < 1$ the scheme produces C^1 limits.

Proof. By Theorem 3.22 it is sufficient to show that the three-scale scheme $\mathbb{S}_Q$ defined by the masks

$$Q_{\beta,1}(z) = \beta(1+z), \qquad Q_{\beta,2}(z) = \frac{1-\beta}{4} z^{-1}(1+z)^3\bigl(1+z^2\bigr),$$

is strongly convergent for the range $0 \leqslant \beta < 1$ and quasi convergent for $-1/3 < \beta < 0$. By Theorem 3.20 this is true if the partial difference scheme $\widetilde{\Delta\mathbb{S}_Q}$ converges to zero for the range $-1/3 < \beta < 1$. The masks of $\widetilde{\Delta\mathbb{S}_Q}$ are given by

$$R_{\beta,1}(z) = \beta z, \qquad R_{\beta,2}(z) = \frac{1-\beta}{4} z^2(1+z)^2.$$

The scheme $\widetilde{\Delta\mathbb{S}_Q}$ is defined by

$$\begin{aligned} f_k^{j+2} &= \bigl(R_{\beta,1} *_1 f^{j+1}\bigr)_k + \bigl(R_{\beta,2} *_2 f^j\bigr)_k \\ &= \begin{cases} \beta f^{j+1}_{(k-1)/2}, & k(\mathrm{mod}\ 2) \equiv 1 \\ 0, & \text{else} \end{cases} + \begin{cases} \frac{1-\beta}{4} f^j_{(k-2)/4}, & k \equiv 2(\mathrm{mod}\ 4), \\ \frac{1-\beta}{2} f^j_{(k-3)/4}, & k \equiv 3(\mathrm{mod}\ 4), \\ \frac{1-\beta}{4} f^j_{(k-4)/4}, & k \equiv 0(\mathrm{mod}\ 4), \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Observe that $\|\widetilde{\Delta\mathbb{S}_Q}\|_\infty < 1$ whenever

$$\max\left(|\beta| + \left|\frac{1-\beta}{2}\right|, \left|\frac{1-\beta}{4}\right|\right) < 1.$$

We see that for the range $-1/3 < \beta < 1$ we have $\|\widetilde{\Delta\mathbb{S}_Q}\|_\infty < 1$. Observe that as $\beta \to 1$ the scheme $\mathbb{S}_\beta$ converges to the interpolatory linear B-spline scheme which is only C^0. On the other hand the solution to

$$\min_{-1<\beta<1} \max\left(|\beta| + \left|\frac{1-\beta}{4}\right|, \left|\frac{1-\beta}{2}\right|\right),$$

is at $\beta = 0$. This corresponds to maximal possible Hölder exponent of the first derivative. We see a tradeoff between ‘near interpolation’ for values of β just below 1 and higher smoothness at $\beta = 0$. Note that for $\beta = 0$ we obtain that

$$P_{\beta,1}(z) = 0, \qquad P_{\beta,2}(z) = \tfrac{1}{16} z^{-4}(1+z)^4\big(1+z^2\big)^2.$$

Thus, the corresponding $\mathbb{S}$-refinable function satisfies a (non-binary) two-scale relation

$$\phi(x) - \sum_{k\in\mathbb{Z}} p_{2,k}\phi(4x-k). \qquad \square$$

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Appendix A

Proof of Lemma 3.17. We use induction. For $m = 1$ the claim is obvious. Assume the claim is true for all $1 \leqslant m' < m$. Since $\xi^{2^{m-1}}$ is a second root of unity there are two possibilities. If $\xi^{2^{m-1}} = -1$ we can chose $n = m-1$. Else we must have $\xi^{2^{m-1}} = 1$ and by induction there exists $0 \leqslant n < m-1$ such that $\xi^{2^n} = -1$. $\square$

Proof of Lemma 3.18 (contributed by Ed Saff). It is sufficient to prove that the complex function $f(z) := \sum_{n=M}^{\infty}(a_{n+1} - a_n)z^n$ is analytic in a neighborhood of the unit. In such a case the sequence $\{a_n\}$ converges since

$$a_N = a_N - a_M + a_M = a_M + \sum_{n=M}^{N-1}(a_{n+1} - a_n) \underset{N\to\infty}{\to} a_M + f(1).$$

We denote

$$g(z) = \sum_{n=M}^{\infty}\left(a_{n+1} - a_n - \sum_{m=1}^{M-1} C_m a_{n-m+1} + \sum_{m=1}^{M-1} C_m a_{n-m}\right) z^n.$$

It is easy to see from the conditions of the lemma that $|g(z)| \leqslant C\sum_{n=M}^{\infty}\mu_n|z|^n$ and thus $g(z)$ is analytic in $|z| < \mu^{-1}$. Also, we have that $g(z) = f(z)q(z)$ where $q(z) := 1 - \sum_{m=1}^{M-1} C_m z^{m-1}$. Since $\sum_{m=1}^{M-1} C_m = 1$, we have that $q(1) = 0$. As we assumed $C_m > 0$ for all $m = 1,\dots,M-1$, by a simple geometric argument, the unit is the only zero of the polynomial q in $|z| \leqslant 1$. Furthermore, using again the positivity of the constants, it can be shown that the multiplicity of the unit as a zero of $q(z)$ is one. Therefore, since $g(z) = f(z)q(z)$ there exists $\varepsilon > 0$ such that

$$f(z) = \frac{A}{z-1} + h(z),$$

with $h(z)$ analytic in $|z| \leqslant 1+\varepsilon$. There are two cases. If $A = 0$, then we are done, since $f(z)$ is analytic in $|z| \leqslant 1+\varepsilon$. Else $A \neq 0$ and in $|z| < 1$ we have that $h(z) = \sum_{n=0}^{\infty} h_n z^n$, and that

$$f(z) = \sum_{n=0}^{\infty}(h_j - A)z^n = \sum_{n=M}^{\infty}(a_{n+1} - a_n)z^n.$$

This implies that the sequence $\{a_n\}$ is not bounded, since for $n \geqslant M$

$$a_{n+1} - a_n = h_n - A \underset{n\to\infty}{\to} -A.$$

But we arrive at a contradiction since the sequence $\{a_n\}$ must be bounded as we show now. To show that $\{a_n\}$ is bounded one can use induction to prove that for $n \geqslant M$ we have

$$|a_n| \leqslant \max_{1 \leqslant j \leqslant M-1} |a_j| + C \sum_{j=M}^{n} \mu^j,$$

which implies that

$$|a_n| \leqslant \max_{1 \leqslant j \leqslant M-1} |a_j| + \frac{C}{1-\mu}.$$

Therefore the constant A must be zero and $f(z)$ is analytic in a neighborhood of $z = 1$. □

Proof of Lemma 3.19. We estimate the differences $|a_{n+1} - a_n|$ by

$$\begin{aligned} |a_{n+1} - a_n| &= \bigl|a_{n+1} - (pa_n + qa_{n-1}) + (pa_n + qa_{n-1}) - a_n\bigr| \\ &\leqslant \bigl|a_{n+1} - (pa_n + qa_{n-1})\bigr| + |q||a_n - a_{n-1}| \\ &\leqslant C\mu^{n+1} + |q||a_n - a_{n-1}| \\ &\leqslant C\mu^{n+1}\left(1 + \frac{|q|}{\mu} + \cdots + \left(\frac{|q|}{\mu}\right)^n\right) \\ &\leqslant \widetilde{C}\bigl(\max(\mu, |q|)\bigr)^{n+1}. \end{aligned}$$

Therefore, since $\max(\mu, |q|) < 1$, the sequence $\{a_n\}$ converges. □

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