Function Spaces associated with non-negative self-adjoint operators

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Some logistics…

- Grade - 75% exam, 25% home assignments (3)
- Exam – Writing the proofs for theorems from a given list.
- The course is based on an upcoming book.

Contents

1.1.1. $\mathbf{M} = \mathbb{T}$. We start with a simple example of a compact manifold with no boundary, the one-dimensional Torus (circle) $M = \mathbb{T} := [-\pi, \pi]$. It is equipped with the measure

$$d\mu(x) := \frac{1}{2\pi} dx,$$

and the 'circle' distance

$$\rho(x, y) = \min_{n \in \mathbb{Z}} |x - y + 2\pi n|, \quad \forall x, y \in \mathbb{T}.$$

In this first example, it is easy to see that the growth condition (1.14) we require in the general case on the volume of balls is satisfied. That is, with $B(x, r) := \{y \in \mathbb{T} : \rho(x, y) < r\}$, we have for any $x \in \mathbb{T}$ and $r > 0$, $|B(x, 2r)| = 2|B(x, r)|$, where $|E|$ is the measure of a measurable set $E \subset \mathbb{T}$.

For $f \in C^2(\mathbb{T})$ we define $Lf := -\Delta f = -f''$. The Laplace operator Δ is at the heart of many important differential equations, in particular, the heat equation we review below. First we observe that the our chosen initial domain $C^2(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ and that $Lf \in L^2(\mathbb{T})$, for any $f \in C^2(\mathbb{T})$. Secondly, that L is a self adjoint operator in the following sense: for any $f, g \in C^2(\mathbb{T})$, we have that

$$\langle Lf, g \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} Lf(x)\overline{g(x)} dx = \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\overline{Lg}(x) dx = \langle f, Lg \rangle.$$

It is also a non-negative operator since for any $f \in C^2(\mathbb{T})$,

$$\langle Lf, f\rangle = -\frac{1}{2\pi}\int_{\mathbb{T}} f''(x)\overline{f(x)}dx = \frac{1}{2\pi}\int_{\mathbb{T}} |f'(x)|^2 dx \geq 0.$$

The eigenfunctions of L are the well-known Fourier orthonormal basis of $L^2(\mathbb{T})$, $\{e^{ik\cdot}\}_{k=\infty}^{\infty}$, with eigenvalues $\{k^2\}_{k=0}^{\infty}$. Observe that the non-negativity of the spectra correlates with the non-negativity of the operator. Observe also that $Le^{ik\cdot} \in C^2(\mathbb{T}) \subset L^2(\mathbb{T})$ for any $k \in \mathbb{Z}$. On the other hand, if we consider $e^{iw\cdot}$, for $w \in \mathbb{R}\setminus\mathbb{Z}$, then it is readily seen that its derivatives are distributions not in $L^2(\mathbb{T})$.

Thus, with the eignestructure at hand, the spectral representation of the operator L is

$$Lf(x) = \sum_{k=-\infty}^{\infty} k^2 \hat{f}(k) e^{ikx}, \quad \hat{f}(k) := \frac{1}{2\pi}\int_{\mathbb{T}} f(u)e^{-iku}du.$$

One can rewrite this representation so that it correlates with the non-negative form (1.49) below

$$L = \sum_{k=1}^{\infty} k^2 dE_k, \quad E_k f(x) := \sum_{j=1}^{k} \left(\hat{f}(-j)e^{-ijx} + \hat{f}(j)e^{ijx} \right).$$

We use **functional calculus** to derive from the spectral representation of the operator L all of the useful representation and approximation tools the reader is possibly familiar with. This is achieved through constructions of useful multipliers on the spectral representation of L. To this end, for an even $\varphi : \mathbb{R} \to \mathbb{R}$ we define the operator $\varphi(L)$, by

$$\varphi(L)f(x) := \sum_{k=-\infty}^{\infty} \varphi(k^2)\hat{f}(k)e^{ikx}. \tag{1.1}$$

For example, the well-known partial Fourier series for any $f \in L^2(\mathbb{T})$ and $N \in \mathbb{N}$, can be represented using $\varphi := \mathbf{1}_{[-1,1]}$

$$\varphi(N^{-1}\sqrt{L})f(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ikx} =: S_N f(x).$$

As is well known, the Fourier orthonormal basis $\{e^{ik\cdot}\}_{k=-\infty}^{\infty}$, is 'optimal' for linear approximation in $L^2(\mathbb{T})$ and provides 'spectral approximation' for the Sobolev spaces $W_2^r(\mathbb{T})$ in the following sense

$$\|f - \varphi(N^{-1}\sqrt{L})f\|_2 \leq N^{-r}\|f^{(r)}\|_2, \quad \forall f \in W_2^r(\mathbb{T}).$$

Theorem Let $f \in W_2^r(\mathbb{T})$ then

$$E_N(f)_2 \le N^{-r} |f|_{r,2} .$$

Proof

1. Decay of the Fourier coefficients - By Parseval, for any $g \in L_2(\mathbb{T})$

$$\|g\|_{L_2(\mathbb{T})}^2 = \sum_{k=-\infty}^{\infty} |\hat{g}(k)|^2 .$$

we have

$$E_N(f)_2 = \|f - S_N(f,x)\|_{L_2(\mathbb{T})} = \sqrt{\sum_{|k| \ge N+1} |\hat{f}(k)|^2}$$

Assume first that $f \in C^r(\mathbb{T})$. We will show $\left|\hat{f}(k)\right| = |k|^{-r}\left|\left(f^{(r)}\right)^{\wedge}(k)\right|$. Using the continuity of f as a periodic function, integration by parts yields,

$$\begin{aligned}\hat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx \\ &= \frac{1}{2\pi}\left(\underbrace{\frac{f(x)e^{-ikx}}{-ik}\bigg|_{-\pi}^{\pi}}_{=0} + \frac{1}{ik}\int_{-\pi}^{\pi} f'(x)e^{-ikx}dx\right) \\ &= \frac{1}{ik}(f')^{\wedge}(k).\end{aligned}$$

By repeated application of the above

$$\left|\hat{f}(k)\right| = |k|^{-r}\left|\left(f^{(r)}\right)^{\wedge}(k)\right|.$$

2. The estimate of the tail

$$
\begin{aligned}
\left\| f(x) - S_N(f,x) \right\|_2^2 &= \sum_{|k| \geq N+1} \left| \hat{f}(k) \right|^2 \\
&\leq N^{-2r} \sum_{|k| \geq N+1} |k|^{2r} \left| \hat{f}(k) \right|^2 \\
&= N^{-2r} \sum_{|k| \geq N+1} \left| \left(f^{(r)} \right)^{\wedge} (k) \right|^2 \\
&\leq N^{-2r} \left\| f^{(r)} \right\|_2^2 .
\end{aligned}
$$

$$
\Rightarrow E_N(f)_2 = \left\| f(x) - S_N(f,x) \right\|_2 \leq N^{-r} \left\| f^r \right\|_2 .
$$

For the general case $f \in W_2^r(\mathbb{T})$ we apply a **density** argument. Let $\{f_j\}_{j=1}^{\infty}$, $f_j \in C^r(\mathbb{T})$, such that

$$\|f - f_j\|_{W_2^r} \underset{j\to\infty}{\to} 0.$$

This implies

$$\|f - f_j\|_2 \underset{j\to\infty}{\to} 0, \quad \|f^{(r)} - f_j^{(r)}\|_2 \underset{j\to\infty}{\to} 0.$$

Therefore

$$\begin{aligned}\|f - S_N(f)\|_2 &\le \|f - f_j\|_2 + \|f_j - S_N(f_j)\|_2 + \|S_N(f_j) - S_N(f)\|_2 \\ &\le N^{-r}\|f_j^{(r)}\|_2 + \|f - f_j\|_2 + \|S_N(f - f_j)\|_2 \\ &\le N^{-r}\|f_j^{(r)}\|_2 + 2\|f - f_j\|_2 \underset{j\to\infty}{\to} N^{-r}\|f^{(r)}\|_2\end{aligned}$$

□

However, it is also well known that if one wishes to approximate in other L^p spaces, $p \neq 2$, one needs other tools. For example, if one wants to ensure convergence of trigonometric approximation of degree N in the L^∞ norm, one may replace the Dirichlet kernel by the Fejér kernel. In the language of functional calculus, the discontinuous $\varphi = \mathbf{1}_{[-1,1]}$ is replaced with the continuous

$$\varphi(\lambda) = \begin{cases} 1 - |\lambda|, & \lambda \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

This yields the trigonometric approximation

$$\varphi(N^{-1}\sqrt{L})f(x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \hat{f}(k)e^{ikx} =: \sigma_N f(x),$$

which satisfies for any $f \in C(\mathbb{T})$

$$\|f - \sigma_N f\|_\infty \underset{N\to\infty}{\longrightarrow} 0.$$

Convolution over the torus

$$f*g(x):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-y)g(y)dy.$$

Def A ***summability kernel*** is a sequence $\{h_N\}$ satisfying:

(i) $\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}h_N(x)dx=1$

(ii) $\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|h_N(x)|dx\le C$.

(ii) For all $0<\delta<\pi$, $\lim\limits_{N\to\infty}\int\limits_{|x|\ge\delta}|h_N(x)|dx=0$

Remark For positive kernels we don't need (ii)

Theorem for a summability kernel $\{h_N\}$ and $f\in C(\mathbb{T})$,

$$\|f-h_N*f\|_{C(\mathbb{T})}=\max_{-\pi\le x\le\pi}|f(x)-h_N*f(x)|\underset{N\to\infty}{\longrightarrow}0.$$

Proof Assume $x = 0$. Let $\varepsilon > 0$. From the uniform continuity of f, there exists $0 < \delta < \pi$, such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

$$h_N * f(0) - f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_N(t)\big(f(-t) - f(0)\big)dt$$

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} h_N(t)\big(f(-t) - f(0)\big)dt + \frac{1}{2\pi}\int_{|x|\geq\delta} h_N(t)\big(f(-t) - f(0)\big)dt$$

$$\left|\frac{1}{2\pi}\int_{-\delta}^{\delta} h_N(t)\left(f(-t)-f(0)\right)dt\right| \le \max_{-\delta\le y\le\delta}\left|f(y)-f(0)\right|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|h_N(t)\right|dt$$

$$\le C\varepsilon.$$

Therefore

$$\left|h_N * f(0)-f(0)\right| \le C\varepsilon + 2\|f\|_\infty \frac{1}{2\pi}\int_{|x|\ge\delta}\left|h_N(t)\right|dt \underset{N\to\infty}{\rightarrow} C\varepsilon .$$

For $x \ne 0$, define $\tilde{f}(t) = f(t+x)$. Then

$$h_N * \tilde{f}(0) = \frac{1}{2\pi}\int \tilde{f}(0-y)h_N(y)dy = \frac{1}{2\pi}\int f(0-y+x)h_N(y)dy$$

$$= \frac{1}{2\pi}\int f(x-y)h_N(y)dy = h_N * f(x).$$

We now apply the first part of the proof for $\tilde{f}$ at 0, observing that $\|\tilde{f}\|_\infty = \|f\|_\infty$ and that for any $\varepsilon > 0$, we can use the same $\delta > 0$ we used for f. Hence, the approximation and convergence are in fact uniform for all $x \in \mathbb{T}$

□

Definition The Fejér kernel of degree $N-1$ is defined by averaging $D_0,\ldots,D_{N-1}$

$$K_N(x) := \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{ikx}$$

$$= \frac{1}{N}\left(N + (N-1)\left(e^{-ix}+e^{ix}\right)+\cdots\right)$$

$$= \sum_{k=-N}^{N}\left(1-\frac{|k|}{N}\right)e^{ikx}$$

$$K_N(x) := \sum_{k=-N}^{N}\left(1-\frac{|k|}{N}\right)e^{ikx} = \frac{1}{N}\left(\frac{\sin(Nx/2)}{\sin(x/2)}\right)^2.$$

The partial Fejér series of f is

$$\sigma_N(f,x) := K_N * f(x) = \sum_{k=-N}^{N}\left(1-\frac{|k|}{N}\right)\hat{f}(k)e^{ikx}.$$

Theorem $\{K_N\}$ is a (positive) summability kernel

Proof

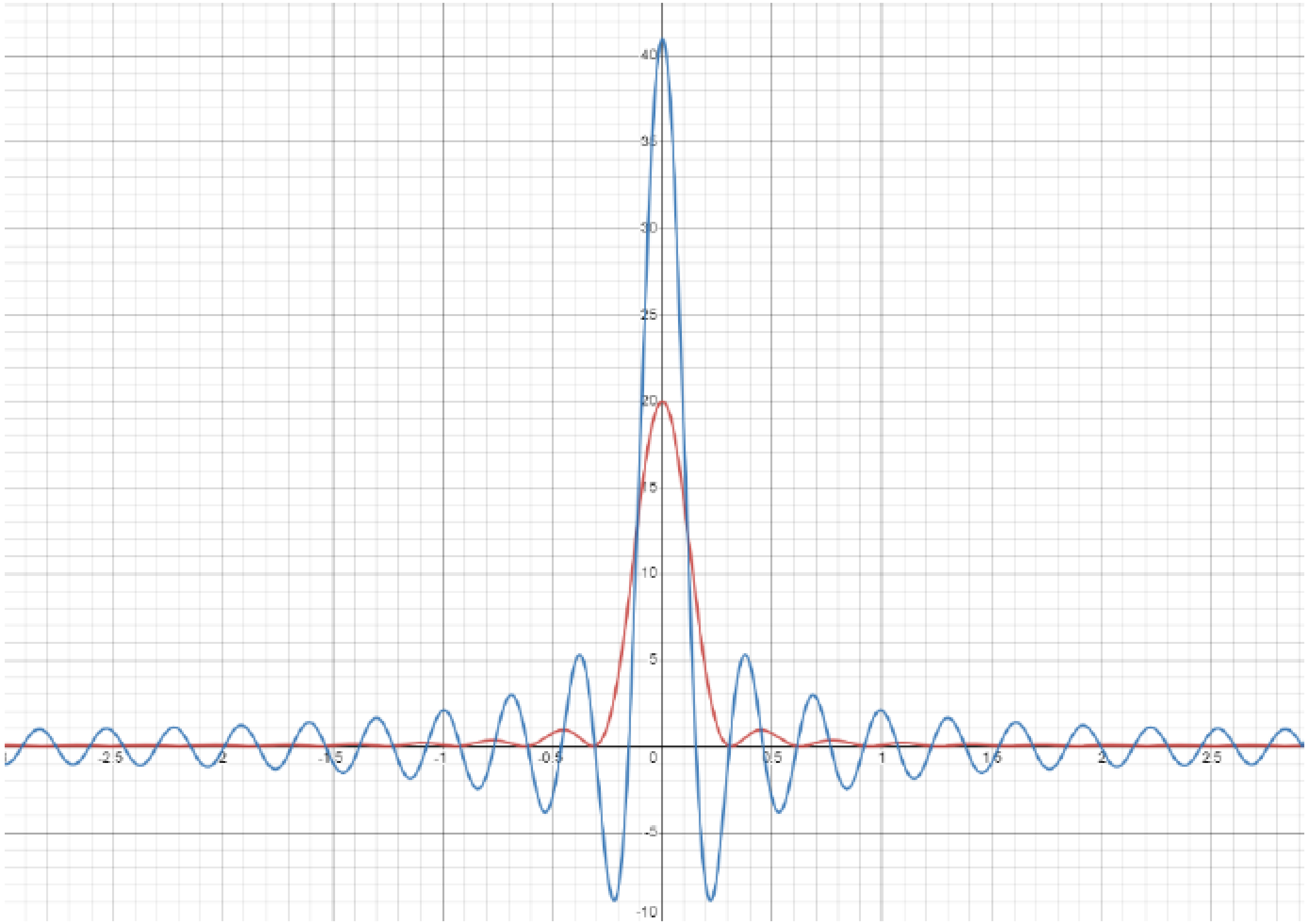
(i) and (ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = \frac{1}{2\pi}\sum_{k=-N}^{N}\left(1-\frac{|k|}{N}\right)\int_{-\pi}^{\pi} e^{ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx = 1$.

(iii) Let $0<\delta<\pi$. Then $\int_{|x|\ge\delta} |K_N(x)|\,dx = \frac{1}{N}\int_{|x|\ge\delta}\left(\frac{\sin(Nx/2)}{\sin(x/2)}\right)^2 dx \le \frac{1}{N}\frac{2\pi}{\sin^2(\delta/2)} \underset{N\to\infty}{\longrightarrow} 0$

Conclusions

(i) $\|f-K_N*f\|_{C(\mathbb{T})} \underset{N\to\infty}{\longrightarrow} 0$.

(ii) The trigonometric polynomials are dense in $C(\mathbb{T})$.



Dirichlet (blue) & Fejér (red) kernels for $N = 20$

In going even further, let $\varphi \in C^\infty(\mathbb{R})$, be even with $\varphi \equiv 1$ on $[-1,1]$, $0 \le \varphi \le 1$, and $\mathrm{supp}(\varphi) \subset [-2,2]$. Then, the operator $\varphi(\delta\sqrt{L})$, $0 < \delta \le 1$, defined by (1.1), has the following good properties:

(ii) It is a kernel operator with a kernel that is rapidly decaying, in the following sense: for any $k \ge 1$, there exists a constant $c_k > 0$, such that

$$|\varphi(\delta\sqrt{L})(x,y)| \le c_k \delta^{-1} \left(1 + \frac{\rho(x,y)}{\delta}\right)^{-k}, \quad \forall x,y \in \mathbb{T}.$$

(iii) For any $r \ge 1$, there exists $c_r > 0$, such that for any $f \in W_p^r$, $1 \le p \le \infty$, and $0 < \delta \le 1$

$$\|f - \varphi(\delta\sqrt{L})f\|_p \le c_r \delta^r \|f^{(r)}\|_p.$$

As we shall see, the ability to construct kernel operators with excellent regularization and localization properties as multipliers through functional calculus, completely generalizes to the most complex cases, e.g. where M is a general Riemannian manifold and L is associated with the natural Laplace-Beltrami operator.

Let us now turn to the discussion of the heat equation. As is well known, the heat equation on $\mathbb{T}$ is

$$\frac{\partial u}{\partial t} = \Delta u, \qquad \Delta u(x,t) := \frac{\partial^2 u}{\partial^2 x}(x,t), \quad \forall x \in \mathbb{T}, t > 0.$$

In this special case where the manifold has no boundaries, one may select $f \in L^2(\mathbb{T})$ as a weak-type initial condition at time $t = 0$, such that $u(x,0) = f(x)$. The solution to the heat equation is generated by the semi-group of operators $\{P_t = e^{-tL}\}_{t>0}$, with heat kernels

$$p_t(x,y) := \sum_{k=-\infty}^{\infty} e^{-tk^2} e^{ik(x-y)}, \quad \forall x, y \in \mathbb{T}. \tag{1.2}$$

That is, for any $f \in L^2(\mathbb{T})$, we have a weak-type solution

$$u(x,t) = P_t f(x) = \frac{1}{2\pi} \int_{\mathbb{T}} p_t(x,y) f(y) dy = \sum^{\infty} e^{-tk^2} \hat{f}(k) e^{ikx}, \quad u(\cdot,0) = f.$$

We would like to show that in this example, the Gaussian upper bound (2.1), which is one of our main assumptions for the general setting, is satisfied. In line with the more general case of the Laplace-Beltrami operator on Riemannian manifold, we also have a lower bound.

PROPOSITION 1.1. There exists $C^\star > 0$, such that for $0 < t \leq 1$, the heat kernel (1.2) satisfies the following lower and upper Gaussian bounds

$$\frac{\sqrt{\pi}}{\sqrt{t}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right) \leq p_t(x,y) \leq \frac{C^\star}{\sqrt{t}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right), \quad \forall x, y \in \mathbb{T}.$$

PROOF. The Fourier transform on $\mathbb{R}$ of

$$g_t(z) := \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-z^2}{4t}\right), \quad t > 0,$$

is

$$\hat{g}_t(\omega) := \exp(-t\omega^2).$$

Plugging this into the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} g_t(z - 2\pi k) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{g}_t(k) e^{ikz},$$

together with (1.2) gives

$$p_t(x,y) = \sqrt{\frac{\pi}{t}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(x-y-2\pi k)^2}{4t}\right). \tag{1.3}$$

For a fixed $x, y \in \mathbb{T}$, let $k_0 \in \mathbb{Z}$, such that $\rho(x,y) = |x-y-2\pi k_0|$. Combining with (1.3) yields the lower bound

$$\sqrt{\frac{\pi}{t}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right) = \sqrt{\frac{\pi}{t}} \exp\left(-\frac{(x-y-2\pi k_0)^2}{4t}\right) \leq p_t(x,y).$$

For the upper bound we use (1.3) and $0 < t \leq 1$ to proceed as follows

$$\begin{aligned} p_t(x,y) &= \sqrt{\frac{\pi}{t}} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(\rho(x,y)-2\pi k)^2}{4t}\right) \\ &\leq \sqrt{\frac{\pi}{t}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right) \left(1 + \sum_{k=-\infty}^{-1} \exp(-\pi^2 k^2) + \sum_{k=1}^{\infty} \exp(1-k)\right) \\ &\leq C^\star \sqrt{\frac{\pi}{t}} \exp\left(-\frac{\rho^2(x,y)}{4t}\right). \end{aligned}$$

□

Finally, it is easy to see, that the heat kernel (1.2), satisfies another main assumption we shall make for the general setting, namely the **Markov property**, also coined **Stochastic Completeness**

$$\frac{1}{2\pi}\int_{\mathbb{T}} p_t(x,y)dy \equiv 1, \quad \forall t > 0.$$

1.1.2. $\mathbf{M} = \mathbb{R}^\mathbf{d}$. We now review our second example for a domain with no boundary, but this time it is not compact, where the space is $M = \mathbb{R}^d$, for some $d \geq 1$. It is equipped with the Lebesgue measure $d\mu(x) = dx$ and the Euclidean distance $\rho(x, y) := |x - y|$. Here, the doubling (growth) condition on volumes of balls (1.14), is satisfied with $|B(x, 2r)| = 2^d |B(x, r)|$, for any $x \in \mathbb{R}^d$ and $r > 0$. We shall see that in the general case, whenever the doubling condition $|B(x, 2r)| \leq c_0 |B(x, r)|$, holds for some $c_0 > 1$, we shall loosely treat $d := \log_2(c_0)$ as the 'dimension' of the corresponding space (see (1.15)).

For any $f \in \mathcal{S}(\mathbb{R}^d)$ we let

$$\Delta f := \sum_{j=1}^{d} \frac{\partial^2 f}{\partial^2 x_j} f,$$

and then we define the self-adjoint non-negative operator $L = -\Delta$. One formal spectral representation that is easy to work with, uses the inverse Fourier transform

$$Lf(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\omega|^2 \hat{f}(\omega) e^{iwx} d\omega, \quad \mathcal{F}(f)(\omega) = \hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-i\omega x} dx.$$

$$\omega x := \langle \omega, x \rangle = \sum_{k=1}^{d} \omega_k x_k$$

With the spectral projections (see §1.5.1)

$$E_\lambda := \mathcal{F}^{-1}\mathbf{1}_{\{|\omega|^2\le\lambda\}}\mathcal{F}, \quad \lambda > 0, \tag{1.4}$$

we have

$$L = \int_0^\infty \lambda dE_\lambda, \tag{1.5}$$

which correlates with the general form for non-negative self adjoint operators (1.49) below. We note in passing, that the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is a dense subdomain of $L^2(\mathbb{R}^d)$, such that $Lf \in L^2(\mathbb{R}^d)$, for any $f \in \mathcal{S}$. Therefore, it may serve as a starting point for the definition of L. As we shall see below in §1.5, in the general case, a non-negative, self-adjoint operator L with the representation (1.5), has the domain

$$D(L) = \{f \in L^2 : \int_0^\infty \lambda^2 d\|E_\lambda f\|_2^2 < \infty\}.$$

In very similar manner to the example of the Fourier series, one can construct an approximation in $L^2(\mathbb{R}^d)$ using band-limited functions through functional calculus. Let $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi := \mathbf{1}_{[-1,1]}$. For any $f \in L^2(\mathbb{R}^d)$ and $\delta > 0$, using (1.4) and (1.5)

$$\begin{aligned}\varphi(\delta\sqrt{L})f &= \int_0^\infty \mathbf{1}_{[-\pi,\pi]}(\delta\sqrt{\lambda})dE_\lambda f\\ &= \mathcal{F}^{-1}\mathbf{1}_{\{|\omega|\le\delta^{-1}\}}\mathcal{F}f.\end{aligned}$$

This implies

$$\lim_{\delta\to 0}\varphi(\delta\sqrt{L})f \underset{L^2}{=} f.$$

Just as with the Fourier series, one has the 'spectral approximation' property for the approximation with band-limited functions. Namely, for any $f \in W_2^r(\mathbb{R}^d)$ and $\delta > 0$

$$\|f - \varphi(\delta\sqrt{L})f\|_2 \le \delta^r |f|_{W_2^r}, \quad |f|_{W_2^r} := \sum_{|\alpha|=r} \|\partial^\alpha f\|_2.$$

Again, this form of approximation is only adequate for L^2 and one needs to apply functional calculus with a smoother cutoff function to approximate in L^p, $1 \le p \le \infty$. A good choice for such a cutoff function is $\varphi \in C_0^\infty(\mathbb{R})$, even, with $\varphi \equiv 1$ on $[-1,1]$, $0 \le \varphi \le 1$, and $\mathrm{supp}(\varphi) \subset [-2,2]$ (see §3.4).

In $M = \mathbb{R}^d$, the heat equation takes the form

$$\frac{\partial u}{\partial t} = \Delta u, \qquad \Delta u(x,t) := \frac{\partial^2 u}{\partial^2 x}(x,t), \quad \forall x \in \mathbb{R}^d, t > 0.$$

Since $\mathbb{R}^d$ is a manifold without boundaries, it suffices to select an initial weak-type condition $f \in L^2(\mathbb{R}^d)$ at time $t = 0$, such that $u(x,0) = f(x)$. The solution to the heat equation is generated by the semi-group of operators $\{P_t = e^{-tL}\}_{t>0}$, with heat kernels

$$p_t(x,y) := \frac{1}{\sqrt{(4\pi t)^d}} \exp\left(-\frac{|x-y|^2}{4t}\right), \quad \forall t > 0. \tag{1.6}$$

That is, for any $f \in L^2(\mathbb{R}^d)$, we have a weak-type solution

$$u(x,t) = P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y) dy.$$

1.1.3. $\mathbf{M = [-1,1]}$. We consider the manifold with a boundary $M = [-1,1]$, its natural distance

$$\rho(x,y) := |\arccos(x) - \arccos(y)|, \tag{1.7}$$

and one of the weighted measures

$$d\mu(x) = \omega_{\alpha,\beta}(x)dx = (1-x)^\alpha(1+x)^\beta dx, \quad \alpha,\beta > -1. \tag{1.8}$$

In this case, verifying the doubling condition (1.14) below is a little less obvious. That is, we need to ensure there exists a constant $c_0 > 1$, such that for any $x \in [-1,1]$ and $r > 0$, $|B(x,2r)| \leq c_0|B(x,r)|$, where by (1.7) and (1.8)

$$|B(x,r)| = \int_{|\arccos(x)-\arccos(y)|<r} (1-y)^\alpha(1+y)^\beta dy.$$

Proposition 1.2. [**10**] There exist $0 < c_1 < c_2 < \infty$, that depend on α and β, such that for any $x \in [-1,1]$ and $0 < r \leq \pi$

$$c_1|B(x,r)| \leq r(\sqrt{1-x^2}+r)^{2\alpha+1}(\sqrt{1+x^2}+r)^{2\beta+1} \leq c_2|B(x,r)|. \tag{1.9}$$

We can now conclude that by (1.9), we get the doubling condition for any $x \in [-1,1]$ and $r > 0$

$$\begin{aligned}|B(x,2r)| &\leq c_1^{-1}2r(\sqrt{1-x^2}+2r)^{2\alpha+1}(\sqrt{1+x^2}+2r)^{2\beta+1}\\ &\leq 2c_1^{-1}\max(1,2^{2\alpha+1})\max(1,2^{2\beta+1})[r(\sqrt{1-x^2}+r)^{2\alpha+1}(\sqrt{1+x^2}+r)^{2\beta+1}]\\ &\leq 2c_1^{-1}c_2\max(1,2^{2\alpha+1})\max(1,2^{2\beta+1})|B(x,r)|.\end{aligned}$$

Next, we let L be the Jacobi operator, initially on functions $f \in C^2[-1,1]$ (observing that $C^2([-1,1])$ is dense in $L^2(\omega)$)

$$(1.10) \qquad Lf(x) := -\frac{[\omega(x)a(x)f'(x)]'}{\omega(x)}, \quad \omega(x) := \omega_{\alpha,\beta}(x), \quad a(x) := 1 - x^2.$$

Since

$$(1.11) \qquad Lf(x) = -(1-x^2)f''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)f'(x),$$

it is easy to see that $Lf \in L^2(\omega)$, for any $f \in C^2[-1,1]$. Also

$$\langle Lf, g \rangle = \int_{-1}^{1} a(x)f'(x)\overline{g'(x)}\omega(x)dx = \langle f, Lg \rangle, \quad \forall f, g \in C^2([-1,1]),$$

which clearly implies that L is a non-negative self adjoint operator. The domain of $\overline{S}$, the closure of $S(f,g) := \langle Lf, g \rangle$ is given by the set of weakly differentiable functions on $(-1,1)$, such that

$$\int_{-1}^{1} |f(x)|^2 \omega(x) dx + \int_{-1}^{1} a(x) |f'(x)|^2 \omega(x) dx < \infty.$$

Lastly, (1.11) implies that $L\mathcal{P}_n \subset \mathcal{P}_n$, where $\mathcal{P}_n$ are the algebraic polynomials of degree n.

The general theorem for non-negative self adjoint operators, Theorem 3.30 below, proves that the spectrum of the Jacobi operators is discrete. Indeed, it is well known that the the normalized Jacobi polynomials $\{P_k\}_{k \geq 0}$ are the eigenfunctions, with $LP_k = \lambda_k$, where $\lambda_k = k(k+\alpha+\beta+1)$. The formal spectral representation of the operator is

$$Lf = \sum_{k=1}^{\infty} \lambda_k \langle f, P_k \rangle P_k.$$

Exactly as in the case $M = \mathbb{T}$, one may apply functional calculus to design kernel operators and specifically kernel operators for polynomial approximation with 'good' properties. The key point is that choosing exactly the same cutoff functions, yields the same outcomes. However, in this example, perhaps the following is not so obvious: Let $\varphi \in C_0^\infty(\mathbb{R})$, be even with $\varphi \equiv 1$ on $[-1,1]$, $0 \le \varphi \le 1$, and $\mathrm{supp}(\varphi) \subset [-2,2]$. Then, the operator $\varphi(\delta\sqrt{L})$, $0 < \delta \le 1$, defined formally by

$$\varphi(\delta\sqrt{L})f = \sum_{k=1}^{\infty} \varphi(\delta\sqrt{\lambda_k})\langle f, P_k\rangle P_k,$$

is a kernel operator with a kernel that is rapidly decaying with respect to the metric (1.7). That is, for any $k \ge 1$, there exists a constant $c_k > 0$, such that for $0 < \delta \le 1$

$$|\varphi(\delta\sqrt{L})(x,y)| \le c_k(|B(x,\delta)||B(y,\delta)|)^{-1/2}\left(1 + \frac{\rho(x,y)}{\delta}\right)^{-k}, \quad \forall x, y \in [-1,1].$$

We now come to the heat kernels $\{e^{-tL}\}_{t>0}$, associated with the Jacobi operator L given in (1.10). Their kernels take the form

$$p_t(x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} P_k(x) P_k(y), \quad \forall x, y \in [-1,1], t > 0.$$

Here, it is not so easy to prove that the heat kernels satisfy the following Gaussian bounds

Proposition 1.3. [**10**] Given the metric (1.7) and the measure (1.8), there exist constants $c_1, C_1, c_2, C_2 > 0$ depending only on α and β, such that for $0 < t \le 1$

$$\frac{C_1 \exp\left(-\frac{c_1 \rho^2(x,y)}{t}\right)}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}} \le p_t(x,y) \le \frac{C_2 \exp\left(-\frac{c_2 \rho^2(x,y)}{t}\right)}{\sqrt{|B(x,\sqrt{t})||B(y,\sqrt{t})|}}, \quad \forall x, y \in [-1,1].$$

1.1.5. Riemannian manifolds with non-negative Ricci curvature. We assume M is a complete, connected, smooth d-dimensional manifold equipped with a Riemannian metric g, which is a symmetric, positive definite, bilinear form on the tangent spaces T_xM, smoothly depending on $x \in M$ [**33**]. For any smooth path $\gamma : (a, b) \to M$, its length is given by

$$\ell(\gamma) = \int_a^b |\dot{\gamma}(t)| dt,$$

where for any tangent vector $\xi \in T_xM$, $|\xi| := \sqrt{\langle \xi, \xi \rangle_g}$. This yields a natural distance for any $x, y \in M$

$$\rho(x, y) := \inf\{\ell(\gamma) : 0 < a < b < \infty, \gamma : (a, b) \to M, \gamma \text{ smooth }, \gamma(a) = x, \gamma(b) = y\}.$$

The topology induced by ρ coincides with the original topology of the smooth manifold. Next, the metric g naturally leads to a canonical measure μ, such that on any chart of the manifold $U \subset \mathbb{R}^d$, in local coordinates, $d\mu = \sqrt{\det g} dx$, where dx is the Lebesgue measure on U.

If the Ricci curvature of the d-dimensional manifold is non-negative, then the Bishop-Gromov comparison theorem (see [**27**, Theorem 3.101]) implies that for any fixed $x \in M$, the function $F_x : \mathbb{R}_+ \to \infty$

$$F_x(r) := \frac{|B(x,r)|}{r^d},$$

is non increasing. This provides a doubling condition, similar to the Euclidean case

$$|B(x,2r)| \leq 2^d |B(x,r)| \quad \forall x \in M, r > 0.$$

However, as an example, for the Hyperbolic space $\mathbb{H}^d$, that has a negative Ricci curvature, we have $|B(x,r)| \sim e^{(d-1)r}$, uniformly for $x \in \mathbb{H}^d$ and so the above doubling condition (which we require in our setup) is not satisfied.

We then have $L = -\Delta$, where $\Delta := \operatorname{div} \circ \nabla$ is the associated Laplace-Beltrami operator. Here, we can also define L initially on $C_0^2(M)$ and it can be shown to be