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Multilevel preconditioning for partition of unity methods: some analytic concepts

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Abstract This paper is concerned with the construction and analysis of multilevel Schwarz preconditioners for partition of unity methods applied to elliptic problems. We show under which conditions on a given multilevel partition of unity hierarchy (MPUM) one even obtains uniformly bounded condition numbers and how to realize such requirements. The main anlytical tools are certain norm equivalences based on two-level splits providing frames that are stable under taking subsets.

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1 Introduction

The so called *meshless methods* are drawing increasing attention in many areas of engineering applications since they avoid notorious difficulties with meshing complicated domains, in particular, when dealing with three or more spatial variables. Meshless methods have come under various names such as "moving least squares", "partition of unity method (PUM)", "radial basis functions", "web splines", "generalized finite elements" or " smoothed particle hydrodynamics". Recent accounts of the state of the art can be found in [1,11], see also the references cited there. There are close conceptual links with more theoretically motivated directions of studies in the group of Triebel (see e.g., [13]) centering on atomic decompositions related to PUM. While most of the numerical work refers to issues like error estimates and functionality of the method, comparatively less seems to be known about fast solution methods for the systems of equations arising from meshless discretization concepts. There is an impressive body of work on *multigrid solvers* for certain variants of PUM documented in [7,8,10,17] which shows very good performance, see also [18] for work on generalized finite elements. On the other hand, it seems that rigorous estimates are still lacking nor is it clear how well these techniques comply with adaptive strategies.

Here we shall focus on the following model problem. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a symmetric bilinear form on a Hilbert space V with norm $\|\cdot\|_V = \langle\cdot,\cdot\rangle^{1/2}$ that is V-elliptic, i.e. there exist positive constants c_a, C_a such that

$$a(v, v) \geq c_a \|v\|_V^2, \qquad |a(v, w)| \leq C_a \|v\|_V \|w\|_V, \quad v, w \in V. \tag{1.1}$$

For any given $f \in V'$ find $u \in V$ such that $a(u, v) = \langle f, v\rangle$, $v \in V$. In what follows V will always be assumed to be one of the spaces $H^1(\Omega)$ or $H_0^1(\Omega)$ corresponding to Neumann or Dirichlet boundary conditions. We shall always assume in what follows that Ω is a bounded *extension* domain. This means that Ω has a sufficiently regular boundary to permit any element v of any Sobolev or Besov space $X(\Omega)$ over Ω to be extended to $\tilde{v} \in X(\mathbb{R}^d)$, $\tilde{v}|_\Omega = v$, in such a way that $\|v\|_{X(\mathbb{R}^d)} \leq C_X \|v\|_{X(\Omega)}$. This is, for instance, the case when the boundary of Ω is piecewise smooth and a uniform cone condition holds for Ω.

The objective of this paper is to develop a multilevel Schwarz preconditioner in the PUM setting that provides even uniformly bounded condition numbers for elliptic boundary value problems. The primary focus of this investigation is a sound theoretical foundation of this issue. Our emphasis here is on bringing out some basic principles that seem to be relevant in such a context and most of the results will be asymptotic in nature. Moreover, it will be seen to comply well with adaptive refinements. Many quantitative aspects such as treating inhomogeneous boundary conditions, dealing with jumping diffusion coefficients or the important issue of quadrature will not be addressed here.

In Sect. 2 we shall describe the general setting of a multilevel covers of Ω on which the construction of multilevel systems of atoms and resulting partition of unity hierarchies (MPUH) in Sect. 3 will be based upon. The central issues in this section are to establish certain scalewise stability properties as well as approximation bounds. The latter estimates as well as certain multilevel representations are based on suitable versions of quasi-interpolants. In particular, we shall identify several conditions, especially concerning certain *local linear independence properties*, that, combined with two-level splits in multilevel expansions, will later be crucial for proving norm equivalences based on these representations in many smoothness spaces. In Sect. 4 we return to problem (1.1) and formulate a multilevel Schwarz preconditioner based on the multilevel representations from the previous section. Moreover, we indicate briefly some possible combination with adaptive solution strategies as well as the relevance of best N-term approximation in this context. The fact that the proposed preconditioner gives rise to uniformly bounded condition numbers is a consequence of the norm equivalences established in Sect. 5.

For the sake of convenience we shall sometimes use the notation $a \lesssim b$ if $a \leq Cb$ with a constant C independent of all parameters on which a, b depend. Similarly, $a \sim b$ means that both relations $a \lesssim b$ and $b \lesssim a$ hold.

2 Discrete multilevel covers of $\Omega \subseteq \mathbb{R}^d$

We wish to discretize (1.1) with the aid of a multilevel partition of unity hierarchy (MPUH) which will be based on certain multilevel covers of the domain Ω. To this end, let $B_r(x)$ denote the (open) ball of radius $r > 0$ and center $x \in \mathbb{R}^d$. We call an open set $\theta \subset \mathbb{R}^d$ a *proper cell* if it has the following properties:

(p1) θ is *star-shaped*, i.e. there exists a “center” x_θ such that for any $x \in \partial\theta$ (the boundary of θ) the line segment $[x_\theta, x]$ connecting x and x_θ is contained in $\bar{\theta}$.

(p2) One can find $r_1 \leq r_2$ such that for a given $R \geq 1$

$$B_{r_1}(x_\theta) \subseteq \theta \subseteq B_{r_2}(x_\theta), \quad \text{where} \quad r_2/r_1 \leq R.$$

Clearly, balls as well as hypercubes are proper cells. Note that proper cells can be dilated. For any positive a let

$$s_a(\theta) := \{x \in \mathbb{R}^d : \exists\, y \in \partial\theta \qquad \text{s.t. } x \in [x_\theta, x_\theta + a(y - x_\theta)]\}. \tag{2.2}$$

For a given compact domain $\Omega \subset \mathbb{R}^d$ (with the properties mentioned in the previous section) or $\Omega = \mathbb{R}^d$, we assume that Θ is a discrete multilevel collection of proper cells in $\mathbb{R}^d$ ($d \geq 1$) of the form

$$\Theta = \bigcup_{m=0}^{\infty} \Theta_m$$

with the following properties: For given positive constants $a_0, a_1, a_2, \ldots$ and N_1 one has:

(C1) For $m \in \mathbb{N}_0$ we have $\Omega \subseteq \bigcup_{\theta \in \Theta_m} \theta$ and $a_1 2^{-a_0 m} \leq |\theta| \leq a_2 2^{-a_0 m}$ for all $\theta \in \Theta_m$, where $|\theta|$ denotes the volume of θ.

(C2) At most N_1 cells from Θ_m may have a nonempty intersection.

(C3) If $\theta \cap \theta' \neq \emptyset, \theta, \theta' \in \Theta_m$, then $|\theta \cap \Omega| \geq a_3|\theta|$ and $|(\theta \backslash \theta') \cap \Omega| \geq a_3|\theta|$.

(C4) For every $x \in \Omega$ and $m \in \mathbb{N}_0$ there exists $\theta \in \Theta_m$ such that $x \in s_{a_4}(\theta)$ for some $a_4 < 1$.

(C5) For all $\theta \in \Theta_m$, $\eta \in \Theta_{m+1}$ we either have $\theta \cap \eta = \emptyset$ or $|\theta \cap \eta \cap \Omega| > a_5|\eta|$.

On account of (C3) and (C5) we shall from now on adopt the convention that θ is always understood to mean $\theta \cap \Omega$.

With any cover of the above type we can associate a parameter vector $\mathbf{p} = \mathbf{p}(\Theta)$ containing all the constants appearing in the above requirements (C1)–(C5) and in properties (p1), (p2) (in fact, we shall extend this list on occasion as we proceed). Note that by (C2) the number of overlaps is controlled, while (C3) says that every two cells from Θ_m are essentially different. (C4) means that every point in the domain is "well covered" by at least one proper cell, while (C5) controls the overlap between cells from two successive levels. Somewhat more can be said.

Remark 2.1 From the definition of a proper cell and (C1) it follows that for any $\theta \in \Theta_m$ we have $\operatorname{diam}\theta \sim 2^{-\tilde{a}_0 m}$ where $\tilde{a}_0 := a_0/d$, with constants of equivalence depending only on $\mathbf{p}(\Theta)$. Moreover, for any $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+1}$ there exist balls

$$B_{r_1}(x_{\theta'}) \subseteq \theta', \quad \theta \subseteq B_{r_2}(x_\theta), \qquad \text{s.t.} \quad r_2/r_1 \le a_6,$$

with a_6 depending only on $\mathbf{p}(\Theta)$.

Of course, thinking of applications where the centers x_θ are given, depending on their distribution, it might be difficult to construct covers with the above properties. When thinking of applications to boundary value problems, one is free to choose centers as well as the shape of cells that accommodate the construction and covers. Note that one typically does not adapt the covers to domain boundaries. The perhaps simplest construction can be sketched as follows. For simplicity let $\Omega = \mathbb{R}^2$ and let the lattice points $k = (k_1, k_2) \in \mathbb{Z}^2$ be the centers at level 0. Let

$$\Theta_m = \{2^{-m}[k_1 - b, k_1 + b] \times [k_2 - b, k_2 + b] := 2^{-m}(k + [-b, b]^2) : k \in \mathbb{Z}^2\}, \tag{2.3}$$

where $b \in (1/2, 1)$ is fixed. Thus $a_0 = 2 = d$, $|\theta| = 2^{-2m}(2b)^2$ for $\theta \in \Theta_m$, and obviously, for $\theta, \theta' \in \Theta_m$, $\theta \cap \theta' \neq \emptyset$ one has $|\theta \cap \theta'| \ge 2^{-2m}(2b-1)^2$. Likewise when $\theta' \in \Theta_{m+1}$, $\theta \in \Theta_m$ have nonempty intersection, one can verify that

$$|\theta \cap \theta'| \ge \begin{cases} 2^{-2m}\left(\frac{3b}{2} - 1\right)^2 & \text{if } 2/3 < b < 1; \\ 2^{-2m}\left(\frac{3b}{2} - \frac{1}{2}\right)^2 & \text{if } 1/2 < b \le 2/3. \end{cases} \tag{2.4}$$

Hence, one has $a_1 = a_2 = (2b)^2$ in (C1), $N_1 = 4$ in (C2). Moreover, note that $|\theta \cap \theta'| \ge (2b - 1/2b)^2|\theta|$, $a_4 = 1/2b$ in (C4), and in (C5) $a_5 = \left(\frac{3}{2} - \frac{1}{b}\right)^2$ when $b > 2/3$, while $a_5 = \left(\frac{3}{2} - \frac{1}{2b}\right)^2$ when $1/2 < b \le 2/3$. Of course, rescalings may be necessary near domain boundaries.

Note that when $b \leq 2/3$ certain intersections of small cells with cells from the previous level in (C5) become empty which accounts for the two cases in (2.4). It is also clear how to extend this to general $d \geq 3$.

Remark 2.2 The above example has an additional property that will be exploited later, namely,

$$\forall\, \theta \in \Theta_m \;\; \exists\, \Omega_\theta \subset \theta \qquad \text{s.t.} \quad \theta' \cap \Omega_\theta = \emptyset \;\; \forall \theta' \in \Theta_m \backslash \{\theta\} \tag{2.5}$$

and

$$|\Omega_\theta| \geq a_6 |\theta|. \tag{2.6}$$

We shall refer to a cover with this property as a *sparse* cover and a_6 will be added to the parameter list $\mathbf{p}(\Theta)$. In the above example we have $a_6 = (1-b)^2$.

An important point about covers of the above type is that the spatial localization offered by moving to higher levels is *isotropic*. The setting presented here may be viewed as a specialization of a more general framework put forward in [3] which aims at capturing also anisotropic features.

Finally it will be convenient to confine the subsequent discussion to the slight further constraint that all proper cells are affine images

$$\theta = A_\theta(\overset{\circ}{\theta}) \tag{2.7}$$

of a single proper *reference cell* $\overset{\circ}{\theta}$ with center 0 and volume $|\overset{\circ}{\theta}| \sim 1$. In the above example the A_θ are just compositions of shifts and dilations.

From now on we shall always assume that Θ satisfies properties (C1)–(C5) for some parameter vector $\mathbf{p}(\Theta)$ as well as that (2.7) holds.

3 Construction of multilevel systems of atoms

We shall always assume that $\phi \in C^r(\mathbb{R}^d)$ is a fixed function supported on the reference cell $\overset{\circ}{\theta}$ with $|\overset{\circ}{\theta}| \sim 1$, having some degree of pointwise smoothness $r \in \mathbb{N}$ (in principle, $r = \infty$ is admissible). Moreover, we require that $\phi(x) > 0$ if $x \in \overset{\circ}{\theta}$.

For any $\theta \in \Theta$ we recall (2.7) and set

$$\phi_\theta := \phi \circ A_\theta^{-1}. \tag{3.1}$$

As in PUM we form partitions of unity by defining for any $m \in \mathbb{N}_0$,

$$\varphi_\theta := \frac{\phi_\theta|_\Omega}{\sum_{\theta' \in \Theta_m} \phi_{\theta'}}, \quad \theta \in \Theta_m, \tag{3.2}$$

where Ω is the domain under consideration. By the properties of ϕ and the cover Θ it follows that

$$0 < c_1 \leq \sum_{\theta \in \Theta_m} \phi_\theta(x) \leq c_2, \quad x \in \Omega, \tag{3.3}$$

where the constants c_1, c_2 depend only on $\mathbf{p}(\Theta)$ amd on ϕ. Consequently, $\sum_{\theta \in \Theta_m} \varphi_\theta(x) = 1$.

Suppose further that $\{P_\beta : |\beta| = \beta_1 + \cdots + \beta_d \leq k-1\}$ is a basis for Π_k the space of all polynomials in d variables of total degree $k-1$, normalized by

$$\|P_\beta \phi\|_{L_\infty(\mathring{\theta})} = 1. \tag{3.4}$$

Then for $\theta \in \Theta$ we let

$$P_{\theta,\beta} := P_\beta \circ A_\theta^{-1}.$$

Remark 3.1 As a consequence of the fact that $|\mathring{\theta}| \sim 1$ we have

$$\|P_\beta \phi\|_{L_p(\mathring{\theta})} \sim \|P_\beta \phi\|_{L_q(\mathring{\theta})}, \quad 0 < p, q \leq \infty, \tag{3.5}$$

with constants of equivalence depending only on p, q, k, and ϕ.

We define

$$\Phi_m := \{P_{\theta,\beta}\varphi_\theta : \theta \in \Theta_m, |\beta| \leq k-1\} \tag{3.6}$$

and set

$$S_m := \text{span}\,(\Phi_m) \quad \text{on } \Omega.$$

Remark 3.2 It is easy to see that for each $m \in \mathbb{N}_0$

$$\Pi_k\,|_\Omega \subset S_m,$$

i.e. for every $P \in \Pi_k$ there exists a $g \in S_m$ such that $P|_\Omega = g$.

3.1 Local linear independence

Our goal is to approximate the solution to (1.1) by linear combinations of the atoms $P_{\theta,\beta}\varphi_\theta$, $\theta \in \Theta$, $|\beta| < k$. This raises a number of well-known practical issues such as the notorious problem of quadrature or the treatment of boundary conditions. In contrast to pure radial basis function approaches the incorporation of essential homogeneous Dirichlet conditions is actually in principle easy and, above all, *local*. In fact, whenever the support of an atom overlaps the boundary one can choose the polynomial factor $P_{\theta,\beta}$ to belong to an ideal whose zero set approximates the corresponding boundary segment. This may even offer better accuracy than common triangular approximations. Since these issues have been addressed elsewhere we concentrate here only on the stability issues related to preconditioning the linear systems resulting from corresponding discretizations.

To this end, it will be important that for each $m \in \mathbb{N}_0$ the collection Φ_m is linearly independent and moreover is stable in L_p. There are several possible ways to go about this. In our approach the notion of *local linear independence* will play a central role. Roughly speaking, it will be seen that whenever we can find within each θ a subset on which the (necessarily finitely many) overlapping atoms are linearly independent, we are able to establish stability. The key property that needs to be satisfied for a given "atom-cover system" (Φ, Θ) can be formulated as follows:

Property (LLIN): *For any $\theta \in \Theta_m$, $m \in \mathbb{N}$, there exists some region $\mathcal{N}_\theta$ containing the center of θ, such that*

$$\sum_{\theta' \supset \mathcal{N}_\theta,\ \theta' \in \Theta_m} \sum_{|\beta|<k} c_{\beta,\theta'} P_{\theta',\beta}(x)\varphi_{\theta'}(x) = 0, \quad x \in \mathcal{N}_\theta \implies c_{\beta,\theta'} = 0, \quad |\beta| < k, \theta' \supset \mathcal{N}_\theta. \tag{3.7}$$

The validity of Property (LLIN) is immediate for sparse covers in the sense of Remark 2.2 (see (2.5)), no matter what the atom system Φ is chosen to be. In fact, taking the region Ω_θ as such a subset $\mathcal{N}_\theta$, linear independence of the atoms on Ω_θ follows from the linear independence of the polynomial factors since φ_θ is constant on Ω_θ.
Sparsely shifted B-splines: A simple concrete example in connection with sparse covers is to employ tensor product B-splines of coordinate degree K and maximal smoothness $K - 1$ shifted on a regular grid in such a way that polynomial regions match for overlapping supports and that the resulting cover is sparse in the sense of Remark 2.2, see also the example following Remark 2.1. For instance, for cardinal B-splines the supports are shifts of $[0, K + 1]^d$ and we could shift on the lattice $L\mathbb{Z}^d$ for some $L \in \mathbb{N}$, $L \leq K$ (but close to K to have only a fixed number of overlaps independent of K).

However, if the cover is not sparse, the verification of (LLIN) is less clear. The difficulty encountered in the more general situation is to handle the possible variety of interactions of overlapping atoms still permitted by conditions (C1)–(C5). We shall frequently use the obvious fact, that (3.7) is equivalent to the analogous relation for φ_θ replaced with $\phi_\theta := \phi \circ A_\theta^{-1}$. Moreover, as a consequence of (p2), (C1), we can find a ball $B \subset \overset{\circ}{\theta}$ such that $B_\theta := A_\theta(B)$ satisfies

$$|B_\theta| \geq a_7|\theta|, \tag{3.8}$$

where $0 < a_7 < 1$ also depends only on $\mathbf{p}(\Theta)$.

We shall discuss now scenarios, where Property (LLIN) can be verified in connection with non-sparse covers.

Radial local polynomial bumps: To describe a second natural scenario (although less favorable regarding quadrature), suppose that $\overset{\circ}{\theta}= B_1(0)$ is the unit ball and $\phi(x) := ((1-|x|^2)_+)^K$, where $x_+ := \max\{0, x\}$ and $K \in \mathbb{N}$ is sufficiently large to be specified later. Thus on θ the function ϕ_θ is a polynomial of degree $2K$.

We shall exploit the fact that the ϕ_θ extend to polynomials $\hat{\phi}_\theta(x) = (1-|A_\theta^{-1}x|^2)^K$ on all of $\mathbb{R}^d$ and that local linear independence of polynomials is equivalent to their (global) linear independence.

Now, given θ, consider a subset $\mathcal{N}$ of θ overlapped by some $\theta_j \in \Theta_m$, $j = 1, \ldots, N$, and set $\hat{\phi}_j := \phi_{\theta_j}$. Moreover, choose $\mathcal{N}$ such that $\mathcal{N}$ is not intersected by any of the zero sets Z_j of $\hat{\phi}_j$, but $\partial\mathcal{N} \cap Z_1$ has nonvanishing $d-1$ dimensional measure. If the atoms $P_{j,\beta}\phi_j$ were linearly dependent over $\mathcal{N}$, we must be able to write for some β

$$P_{1,\beta}\hat{\phi}_1 = \sum_{j=2,|\beta|<k}^{N} c_{j,\beta} P_{j,\beta}\hat{\phi}_j. \tag{3.9}$$

Note that the left hand side has a zero of order $2K-1$ on Z_1. Thus the left hand side belongs to the ideal generated the zero set of $\hat{\phi}_1$. Since all the remaining $\hat{\phi}_j$s have no common zero in $\mathbb{C}^n$, the Hilbert Nullstellensatz implies that the ideal generated by these polynomials is the whole ring of polynomials. Thus, as long as $k < K$, say, the right hand side of (3.9) is not expected to be able to produce a zero of order $2K-1$ on $Z_1 \cap \partial\mathcal{N}$. We leave this as an open problem.

Variable order radial polynomial bumps: We describe next a simple way of ensuring local linear independence by slightly extending the setting. This will allow us later to conveniently deduce further properties needed in the multilevel context. Note first that it is not necessary to work with a single bump ϕ. All arguments carry over to a setting where we choose a finite fixed collection of generating bumps ϕ^ν and we restrict the formal discussion otherwise to a single generator only in order to simplify notation.

Before going into technical details we sketch the idea. As indicated before, instead of taking affine compositions of a *single* ϕ as above, we employ a fixed finite number of bumps

$$\phi^\nu(x) := \left((1-|x|^2)_+\right)^{K_\nu}, \quad \nu = 1, \ldots, 2N_1, \quad x_+ := \max\{x, 0\},$$

where the choice of the parameters K_ν will be explained in a moment. This additional flexibility will allow us to dispense with the sparse covering property. What remains important is that at most a controled number N_1 of atoms overlap at a given point. Then it is possible to color the elements of any two successive levels Θ_m, Θ_{m+1} by at most $2N_1$ colors in such a way that any two θ of the same color are disjoint (the interaction of two successive levels will be important later). Given a fixed numbering of these colors and using a fixed polynomial order k of the polynomial factors P_β, we choose now $K_{\nu+1} > k + K_\nu$, $\nu = 1, \ldots, 2N_1 - 1$. Thus whenever the supports of a set of atoms have a nonempty intersection, these atoms will have strictlly distinct polynomial degrees on this intersection. From this it is then easy to see that the atoms are everywhere locally and therefore also globally linearly independent.

To make this more precise, we begin by splitting the ellipsoid cover Θ into no more than $2N_1$ disjoint subsets (colors) $\{\Theta^\nu\}_{\nu=1}^{2N_1}$ so that neither two ellipsoids $\theta', \theta'' \in \Theta_m \cup \Theta_{m+1}$ with $\theta' \cap \theta'' \neq \emptyset$ are of the same color. Indeed, using property (C2) of Θ it is easy to color any level Θ_m by using no more than N_1 colors. So, we use at most N_1 colors to color the ellipsoids in $\{\Theta_{2j}\}_{j\in\mathbb{Z}}$ and further at most N_1 colors to color the ellipsoids in $\{\Theta_{2j+1}\}_{j\in\mathbb{Z}}$. Thus we may assume that we have the following disjoint splitting

$$\Theta = \bigcup_{\nu=1}^{2N_1} \Theta^\nu \quad \text{and} \quad \Theta_{2j} = \bigcup_{\nu=1}^{N_1} \Theta_{2j}^\nu, \quad \Theta_{2j+1} = \bigcup_{\nu=N_1+1}^{2N_1} \Theta_{2j+1}^\nu, \quad j \in \mathbb{N}_0, \tag{3.10}$$

where if $\theta' \in \Theta_{m_1}^{\nu_1}$, $\theta'' \in \Theta_{m_2}^{\nu_2}$ with $|m_1 - m_2| \leq 1$, and $\theta' \cap \theta'' \neq \emptyset$, then $\nu_1 \neq \nu_2$.

We introduce now $2N_1$ smooth piecewise polynomial bumps associated with the colors from above setting for fixed positive integers M and k ($M \geq k$)

$$\phi^\nu(x) := \left((1 - |x|^2)_+\right)^{M+\nu k}, \quad \nu = 1, 2, \ldots, 2N_1. \tag{3.11}$$

Notice that $\phi^\nu \in C^{M+\nu k-1} \subset C^M$.

For any $\theta \in \Theta$ we then define for each color ϕ_θ and φ_θ exactly as in (3.1), (3.2), arriving again at (3.3).

By construction, only atoms from different colors can overlap a common region. Therefore, when using different generating bumps ϕ^ν, the fact that requirement (3.7) in property (LLIN) is indeed true is a consequence of the fact that the term with the highest ν is of polynomial degree strictly larger than the degrees of all other terms. Thus, peeling off step by step the highest degree terms, confirms (3.7).

3.2 Levelwise stability

In the following we shall briefly write

$$\|g\|_2 = \|g\|_{L_2(\Omega)},$$

whenever the domain under consideration is Ω. The first essential building block is the following *levelwise stability* of the partitions of unity.

Theorem 3.3 *Suppose that Property* (*LLIN*) *is valid. Then each collection* Φ_m ($m \in \mathbb{N}_0$) *is linearly independent on* Ω *and hence forms a basis for* $S_m := \operatorname{span}(\Phi_m)$. *Moreover, any* $g \in S_m$ *has a unique representation*

$$g = \sum_{\theta \in \Theta_m,\ |\beta| < k} b_{\theta,\beta}(g) P_{\theta,\beta} \varphi_\theta, \tag{3.12}$$

where the dual functionals $b_{\theta,\beta}$ can be defined as follows. For every θ there exists B_θ with $|B_\theta| \sim |\theta|$ such that

$$b_{\theta,\beta}(f) = \langle f, \tilde{g}_{\theta,\beta}\rangle, \quad \textit{where } \operatorname{supp}(\tilde{g}_{\theta,\beta}) \subseteq B_\theta, \quad \|\tilde{g}_{\theta,\beta}\|_\infty \lesssim 1/|\theta|. \tag{3.13}$$

As a consequence we have

$$|b_{\theta,\beta}(g)| \lesssim |\theta|^{-1/2}\|g\|_{L_2(B_\theta)} \quad \forall g \in S_m. \tag{3.14}$$

Moreover, for $g \in S_m$, we have

$$\|g\|_2 \sim \left(\sum_{\theta\in\Theta_m,\ |\beta|<k} \|b_{\theta,\beta}(g)P_{\theta,\beta}\varphi_\theta\|_2^2\right)^{1/2}. \tag{3.15}$$

Here all constants depend only on k, $\mathbf{p}(\Theta)$, ϕ but not on m and g.

Proof In order to reduce technicalities we shall carry out the proof only for the case of a single generating bump ϕ. It is evident how to extend the argument to the more general case of multiple colors.

We shall construct suitable dual functionals by biorthogonalizing local restrictions of interacting atoms. To control the spectrum of the corresponding Gramians we need some preparatory steps. The first one concerns the mutual overlap of atoms from one level. A simple *shrinking argument* combined with (C2) will allow us to identify always *substantial* regions of mutual overlap. □

Lemma 3.4 *For any $\theta \in \Theta_m$ there exists an ellipsoid $B_\theta \subset \theta$ with the property that*

$$B_\theta \cap \theta' \neq \emptyset, \quad \theta, \theta' \in \Theta_m \quad \Longrightarrow \quad |B_\theta \cap \theta'| \geq b_4|\theta|, \tag{3.16}$$

where the constant b_4 depends only on $\mathbf{p}(\Theta)$.

Proof Recall from property (p2) there exists a ball $B_{\bar{\rho}} \subseteq \mathring{\theta}$ such that

$$|B_{\bar{\rho}}| \geq b_1|\mathring{\theta}| \tag{3.17}$$

for some positive constant $b_1 < 1$, where $\bar{\rho}$, b_1 depend only on the constants r_1, R in (p2) and on the fixed reference cell $\mathring{\theta}$. Consider the dilated versions $B^\ell := B_{(1-\frac{\ell}{2N_1})\bar{\rho}}$ of $B_{\bar{\rho}}$ from (3.17), i.e. $B^0 = B_{\bar{\rho}}$ and $B^{N_1} = B_{\bar{\rho}/2}$. Likewise let $B_{\theta,\ell} := A_\theta(B^\ell)$. Thus, by (p2), we have

$$|B_{\theta,\ell}| \geq b_2|\theta|, \quad \ell = 0, \dots, N_1, \tag{3.18}$$

for some uniform constant $b_2 > 0$ depending only on $\mathbf{p}(\Theta)$. Furthermore, note that, again by (p2),

$$\theta' \cap B_{\theta,\ell} \neq \emptyset, \quad \theta' \in \Theta_m \implies |\theta' \cap B_{\theta,\ell-1}| \geq b_3|\theta|, \quad \ell = 1, \ldots, N_1, \tag{3.19}$$

where $b_3 > 0$ is another uniform constant depending only on $\mathbf{p}(\Theta)$.

Next observe that there exists an $\ell^* \in \{1, \ldots, N_1\}$ such that

$$\begin{gathered} B_{\theta,\ell^*} \cap \theta' = \emptyset, \quad \forall \theta' \in \Theta_m \backslash \{\theta\}, \\ \text{or} \\ \text{if } \theta' \in \Theta_m, \quad \theta' \cap B_{\theta,\ell^*-1} \neq \emptyset \implies \theta' \cap B_{\theta,\ell^*} \neq \emptyset. \end{gathered} \tag{3.20}$$

In fact, let $\Xi_\ell := \{\theta' \in \Theta_m : \theta' \neq \theta,\ \theta' \cap B_{\theta,\ell} \neq \emptyset\}$. Clearly $\#\Xi_0 \leq N_1$ (see (C2). If Ξ_1 is empty, we set $\ell^* = 1$. If $\#\Xi_1 = \#\Xi_0$ we again set $\ell^* = 1$ and are done. So, it remains to consider the case $\#\Xi_0 > \#\Xi_1 > 0$. Thus, in general, either (3.20) holds for ℓ or $\#\Xi_{\ell+1} < \#\Xi_l$, so that (3.20) holds after at most N_1 steps. We take now ℓ^* as the smallest integer for which (3.20) is valid and set $B := B_{\ell^*-1}$ when the second case in (3.20) holds or $B := B_{\ell^*}$ when the first case is true. Thus, in summary $B_\theta := A_\theta(B)$ for this B satisfies (3.16). □

Now set $\Gamma_\theta := \{\theta' : \theta' \in \Theta_m, \theta' \cap B_\theta \neq \emptyset\}$ and let

$$\mathcal{C}_\theta := \{g_{\theta',\beta'} := P_{\theta',\beta'}\varphi_{\theta'}\chi_{B_\theta} : \theta' \in \Gamma_\theta,\ |\beta'| < k\},$$

be the collection of all mth level atoms that overlap B_θ (including those corresponding to θ itself). Note that the $g_{\theta',\beta'}$ are defined on all of Ω but vanish outside B_θ. By property (C2), the cardinality of $\mathcal{C}_\theta$ is uniformly bounded by a constant multiple of $N_1 k^d$.

Now consider the local Gramian

$$G_\theta := \left(\langle g_{\theta',\beta'}, g_{\theta'',\beta''}\rangle_{B_\theta}\right)_{\theta',\theta''\in\Gamma_\theta,\,|\beta'|,|\beta''|<k},$$

where $\langle v, w\rangle_{B_\theta} := \int_{B_\theta} vw\,dx$. We shall next show that G_θ is nonsingular and can be used to construct a suitable collection of dual functionals. To this end, note that straightforward substitution yields

$$\begin{aligned} \langle g_{\theta',\beta'}, g_{\theta'',\beta''}\rangle_{B_\theta} &= |A_\theta| \int_B \frac{P_{\beta'}(A_{\theta'}^{-1}A_\theta y)\phi(A_{\theta'}^{-1}A_\theta y)P_{\beta''}(A_{\theta''}^{-1}A_\theta y)\phi(A_{\theta''}^{-1}A_\theta y)}{\left(\sum_{\zeta\in\Gamma_\theta}\phi(A_\zeta^{-1}A_\theta y)\right)^2}dy \\ &= |A_\theta| \int_B \frac{P_\beta(A_{\theta',\theta}y)\phi(A_{\theta',\theta}y)P_{\beta''}(A_{\theta'',\theta}y)\phi(A_{\theta'',\theta}y)}{\left(\sum_{\zeta\in\Gamma_\theta}\phi(A_{\zeta,\theta}y)\right)^2}dy, \end{aligned} \tag{3.21}$$

where $A_{\theta',\theta} := A_{\theta'}^{-1}A_\theta$, $A_{\theta'',\theta} := A_{\theta''}^{-1}A_\theta$ are affine mappings that will be seen next to belong to some compact set independent of $\theta \in \Theta$.

In fact, setting $A_\theta y = M_\theta y + x_\theta$, where x_θ is the center of θ and M_θ is the corresponding $(d \times d)$-matrix, one obviously has

$$A_{\theta'}^{-1}A_\theta y = (M_{\theta'}^{-1}M_\theta)y + M_{\theta'}^{-1}(x_\theta - x_{\theta'}).$$

From property (p2) one infers that

$$|M_{\theta'}^{-1}(x_\theta - x_{\theta'})| \le C, \quad \theta' \in \Gamma_\theta, \tag{3.22}$$

where the constant depends only on $\mathbf{p}(\Theta)$.

Furthermore, considering the singular value decomposition $M_{\theta'}^{-1} M_\theta = U \Sigma V$, U, V orthogonal matrices, the singular values on the diagonal of Σ are contained, on account of property (p2), in a fixed interval $[a_{10}, a_{11}]$ depending only on $\mathbf{p}(\Theta)$ and k, where $a_{10} > 0, a_{11} < \infty$. The orthogonal matrices U, V can also be viewed as elements of a compact finite dimensional manifold. Defining the collection of all affine maps

$$\mathcal{A}(\rho_1, \rho_2) := \{A : Ax = Mx + b,\ B_{\rho_1}(0) \subset M(B_1(0)) \subset B_{\rho_2}(0)\},$$

we see that the mappings $A_{\theta'',\theta}$ all belong to some $\mathcal{A}(\rho_1, \rho_2)$, where ρ_1, ρ_2 depend only on $\mathbf{p}(\Theta)$ but not on θ. Moreover the $A_{\theta',\theta}$ are instances of elements in $\mathcal{A}(\rho_1, \rho_2)$ that can be parametrized over some fixed bounded set $\mathcal{K}$ of finitely many parameters. On account of (3.19) and (3.16) $\mathcal{K}$ is also closed and hence compact. Hence the Gramian can be viewed as a function of the parameters in $\mathcal{K}$ which depends only on $\mathbf{p}(\Theta)$. By (3.3) this dependence is continuous. Therefore, each

$$\tilde{G}_\theta := |A_\theta|^{-1} G_\theta \tag{3.23}$$

can be viewed as the value of a continuous matrix valued function at some point in the compact set $\mathcal{K}$. Thus, by (3.8), the L_∞-norms of the restrictions $g_{\theta',\beta} \in \mathcal{C}_\theta$ are uniformly bounded from above and away from zero,

$$\|g_{\theta',\beta}\|_\infty \sim 1, \tag{3.24}$$

with constants depending on the parameters in $\mathbf{p}(\Theta)$ and ϕ.

The determinant of $\tilde{G}_\theta$ is also the evaluation of a continuous function on $\mathcal{K}$. Note also that the size of $\tilde{G}_\theta$ may vary between $\dim \mathbb{P}_k$ and $N_1 \dim \mathbb{P}_k$. By (3.16) the size remains unchanged under varying the parameters in $\mathcal{K}$. Now, by Property (LLIN) the elements of $\mathcal{C}_\theta$ are linearly independent over some region $\mathcal{N}_\theta$ containing the center of θ. Hence this region intersects B_θ. A simple peeling off argument combined with gradually expanding $\mathcal{N}_\theta \cap B_\theta$ shows that the elements of $\mathcal{C}_\theta$ are still locally linearly independent over B_θ. So, the Gramians are always nonsingular and hence their determinants do not vanish in $\mathcal{K}$. Since $\mathcal{K}$ is compact they attain their minimum in $\mathcal{K}$ that is bounded away from zero from below by some positive constant b_4 depending, in view of (3.16), only on $\mathbf{p}(\Theta)$, k and ϕ. Therefore the inverse $\tilde{G}_\theta^{-1}$ exists and is the value of a continuous function on $\mathcal{K}$ as well. Hence, by the previous remarks and (3.24), we also have

$$\left|(\tilde{G}_\theta^{-1})_{(\theta',\beta'),(\theta'',\beta'')}\right| \lesssim 1,\ (\theta',\beta'), (\theta'',\beta'') \in \Gamma_\theta^k := \Gamma_\theta \times \{\beta \in \mathbb{Z}_+^d : |\beta| < k\}, \tag{3.25}$$

with a constant depending only on $\mathbf{p}(\Theta)$, k and ϕ. Let us denote the entries of the inverse $G_\theta^{-1} = |A_\theta|^{-1}\tilde{G}_\theta^{-1}$ by $R_{(\theta',\beta'),(\theta'',\beta'')}$, $(\theta',\beta'),(\theta'',\beta'') \in \Gamma_\theta^k := \Gamma_\theta \times \{\beta \in \mathbb{Z}_+^d : |\beta| < k\}$. Then the functions

$$\tilde{g}_{\theta,\beta} := \sum_{(\theta',\beta')\in\Gamma_\theta^k} R_{(\theta,\beta),(\theta',\beta')} g_{\theta',\beta'} \tag{3.26}$$

which, by construction, are supported on B_θ, form a dual system to Φ_m. In fact,

$$\begin{aligned}\langle \tilde{g}_{\theta,\beta}, g_{\beta^*,\theta^*}\rangle_{B_\theta} &= \sum_{(\theta',\beta')\in\Gamma_\theta^k} R_{(\theta,\beta),(\theta',\beta')}\langle g_{\theta',\beta'}, g_{\theta^*,\beta^*}\rangle_{B_\theta}\\ &= (G_\theta G_\theta^{-1})_{(\theta,\beta),(\theta^*,\beta^*)} = \delta_{(\theta,\beta),(\theta^*,\beta^*)},\\ &\quad (\theta,\beta),(\theta^*,\beta^*) \in \Gamma_\theta.\end{aligned} \tag{3.27}$$

It remains to prove that the functionals $b_{\theta,\beta}(g) := \langle \tilde{g}_{\theta,\beta}, g\rangle_{B_\theta}$ satisfy (3.13) with constant C depending only on the parameters in $\mathbf{p}(\Theta)$, k and ϕ. Since $|A_\theta| \sim |\theta|$ it immediately follows from (3.23) and (3.25) that

$$|R_{(\theta,\beta),(\theta',\beta')}| \lesssim 1/|\theta|, \tag{3.28}$$

where again the constant depends only on $\mathbf{p}(\Theta)$ and the order k of the polynomials. In view of (3.26), the uniform bound (3.13) follows indeed from (3.28), (3.26) and the fact that $\#\Gamma_\theta \le N_1$, $\theta \in \Theta_m$, $m \in \mathbb{N}_0$.

This finishes the proof for the case of a single generating bump ϕ. It is now evident how to extend the compactness argument to the case of several colored generating bumps. □

3.3 Quasi-interpolants

The second crucial ingredient are Quasi-interpolants mapping $L_2(\Omega)$ onto the spaces S_m. Specifically, the mappings

$$Q_m f := \sum_{\theta\in\Theta_m,|\beta|<k} b_{\theta,\beta}(f) P_{\theta,\beta}\varphi_\theta, \quad f \in L_2(\Omega), \tag{3.29}$$

are, in view of Theorem 3.3, especially (3.14), uniformly bounded projectors from $L_2(\Omega)$ onto S_m.

Lemma 3.5 *We have*

$$\|Q_m f\|_{L_2(\theta)} \le c\|f\|_{L_2(\theta^*)} \quad \forall f \in L_2(\Omega), \tag{3.30}$$

where for $\theta \in \Theta_m$

$$\theta^* := \bigcup\{\theta' \in \Theta_m : \theta \cap \theta' \ne \emptyset\}.$$

Further immediate consequences of Theorem 3.3 concern the approximation properties of the spaces S_m. To this end, consider the usual forward difference of f in direction h defined by $\Delta_h f(x) := \Delta_h^1 f(x) := f(x+h) - f(x)$ when the line segment $[x, x+h]$ is contained in Ω and by $\Delta_h f(x) = 0$ otherwise. Likewise define for $k > 1$ the kth order forward difference by $\Delta_h^k f(x) := \Delta_h(\Delta_h^{k-1} f(x))$, again provided that $[x, x+kh] \subset \Omega$, while $\Delta_h^k f(x) := 0$ otherwise. Recall that the two versions of the kth L_2-modulus of smoothness are then as usual defined as

$$\omega_k(f,\theta)_2 := \sup_{t>0} \sup_{|h|\le t} \|\Delta_h^k f\|_{L_2(\theta)}, \quad \omega_k(f,t)_2 := \sup_{|h|\le t} \|\Delta_h^k f\|_{L_2(\Omega)}.$$

Lemma 3.6 *For $f \in L_2(\Omega)$ and $\theta \in \Theta_m$ one has*

$$\|f - Q_m f\|_{L_2(\theta)} \le c \sum_{\theta' \in \Theta_m : \theta' \cap \theta \ne \emptyset} \omega_k(f,\theta')_2. \tag{3.31}$$

Moreover,

$$\|f - Q_m f\|_{L_2(\Omega)} \le c \left(\sum_{\theta \in \Theta_m} \omega_k(f,\theta)_2^2 \right)^{1/2} \le c\omega_k(f, 2^{-a_0 m/d})_2. \tag{3.32}$$

Hence, one has

$$\|f - Q_m f\|_{L_2(\Omega)} \to 0 \quad \text{as } m \to \infty. \tag{3.33}$$

In addition, denoting by $|f|^2_{H^k(\Omega)} := \sum_{|\beta|=k} \|\partial^\beta f\|^2_{L_2(\Omega)}$, the classical kth order Sobolev semi-norm in L_2, an immediate consequence of (3.31) *is*

$$\|f - Q_m f\|_{L_2(\Omega)} \le c h_m^r |f|_{H^r(\Omega)}, \quad r \le k, \tag{3.34}$$

where $h_m = \max\{\operatorname{diam}\theta : \theta \in \Theta_m\}$. The constants in (3.31)–(3.34) *depend only on $\phi, k, \mathbf{p}(\Theta)$ but not on f, m, θ.*

Proof Estimate (3.31) is an immediate consequence of the locality of the dual functionals, the polynomial reproduction property from Remark 3.2, and a classical Whitney estimate for local polynomial approximation. As for (3.33), it is easy to see (cf. [3, 12]) that

$$\omega_k(f, 2^{-a_0 m/d})_2 \sim \left(\sum_{\theta \in \Theta_m} \omega_k(f,\theta)_2^2 \right)^{1/2}, \tag{3.35}$$

so that (3.33) follows from (3.31) and (3.35). Estimate (3.34) follows from standard estimates for the modulus of smoothness given enough smoothness. □

3.4 Two-level splits

For Schwarz type preconditioners to produce uniformly bounded condition numbers one needs to have stable splittings of the corresponding energy space which, in turn, could be viewed as constructing suitable frames for this space, see e.g., [9,16]. For such multilevel frames to exist one needs to capture in some sense difference information between successive levels of resolution. In the present framework of MPUHs we cannot expect any nestedness of the spaces S_m. Nevertheless, we shall see in this section that appropriate two-level splits can serve to some extent as substitutes.

To describe such two-level splits let

$$\Lambda_m := \{\lambda = (\eta, \theta, \beta) : \eta \in \Theta_{m+1}, \theta \in \Theta_m, |\theta \cap \eta| \neq 0, |\beta| < k\}, \; m \geq 0, \tag{3.36}$$

and define

$$F_\lambda := P_{\eta,\beta}\varphi_\eta\varphi_\theta, \quad \lambda = (\eta, \theta, \beta) \in \Lambda_m. \tag{3.37}$$

Note that

$$\sum_{\eta\in\Theta_{m+1}} \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} \varphi_\eta\varphi_\theta = 1 \quad \text{on } \Omega. \tag{3.38}$$

In order to obtain multilevel decompositions of function spaces based on Θ and the above atoms we shall employ the following two-scale relations of polynomial bases

$$P_{\theta,\alpha} = \sum_{|\beta|<k} m^{\theta,\eta}_{\beta,\alpha} P_{\eta,\beta}.$$

Combining this with the partition of unity property of the $\varphi_\eta \in \Theta_{m+1}$, we obtain for $\theta \in \Theta_m$

$$P_{\theta,\alpha} = \sum_{\eta\in\Theta_{m+1}:\theta\cap\eta\neq\emptyset} \sum_{|\beta|<k} m^{\theta,\eta}_{\beta,\alpha} P_{\eta,\beta}\varphi_\eta. \tag{3.39}$$

Finally it will be convenient to introduce in addition $\Lambda_{-1} := \Theta_0$ to set $\Lambda := \bigcup_{m=-1}^{\infty} \Lambda_m$, and use the same notation for the coarse single-level atoms $F_\lambda := P_{\theta,\beta}\varphi_\theta$, $\lambda = (\theta, \beta) \in \Lambda_{-1}$.

Theorem 3.7 *For any $f \in L_2(\Omega)$ we have (with $Q_{-1} \equiv 0$)*

$$f = \sum_{m=-1}^{\infty} (Q_{m+1}f - Q_m f) = \sum_{m=-1}^{\infty} \sum_{\lambda\in\Lambda_m} d_\lambda(f)F_\lambda = \sum_{\lambda\in\Lambda} d_\lambda(f)F_\lambda, \tag{3.40}$$

where for $m^{\theta,\eta}_{\beta,\alpha}$ *from* (3.39) *and the dual functionals* $b_{\eta,\beta}(\cdot)$ *constructed in Theorem 3.3 one has*

$$d_\lambda(f) = b_{\eta,\beta}(f) - \sum_{|\alpha|<k} m^{\theta,\eta}_{\beta,\alpha} b_{\theta,\alpha}(f). \tag{3.41}$$

Proof The representation (3.40), i.e. the strong convergence of the underlying expansion follows from (3.33). Furthermore, we have

$$\begin{aligned}
Q_{m+1}f - Q_m f &= \sum_{\eta\in\Theta_{m+1}} \sum_{|\beta|<k} b_{\eta,\beta}(f) P_{\eta,\beta}\varphi_\eta - \sum_{\theta\in\Theta_m} \sum_{|\alpha|<k} b_{\theta,\alpha}(f) P_{\theta,\alpha}\varphi_\theta \\
&= \sum_{\theta\in\Theta_m} \varphi_\theta \sum_{\eta\in\Theta_{m+1}} \left(\sum_{|\beta|<k} b_{\eta,\beta}(f) P_{\eta,\beta} \right) \varphi_\eta \\
&\quad - \sum_{\theta\in\Theta_m} \left(\sum_{|\alpha|<k} b_{\theta,\alpha}(f) \sum_{\eta\in\Theta_{m+1}:\theta\cap\eta\neq\emptyset} \sum_{|\beta|<k} m^{\theta,\eta}_{\beta,\alpha} P_{\eta,\beta}\varphi_\theta\varphi_\eta \right) \\
&= \sum_{\eta\in\Theta_{m+1}} \sum_{\theta\in\Theta_m:\ \theta\cap\eta\neq\emptyset} \sum_{|\beta|<k} \left\{ b_{\eta,\beta}(f) - \sum_{|\alpha|<k} m^{\theta,\eta}_{\beta,\alpha} b_{\theta,\alpha}(f) \right\} \\
&\quad \times P_{\eta,\beta}\varphi_\eta\varphi_\theta,
\end{aligned}$$

as claimed. □

For $\lambda = (\eta, \theta, \beta) \in \Lambda_m$ we shall often write $\eta_\lambda = \eta$, $\theta_\lambda = \theta$ and $\beta_\lambda = \beta$.

An important point for later developments is the fact that the representations of the differences $(Q_{m+1} - Q_m)f$ are under certain conditions unique and stable. For this purpose the product atoms F_λ should also be locally linearly independent. We formalize this requirement as follows:

Property (LLIN′): *For each* $\eta \in \Theta_{m+1}$, $m \in \mathbb{N}_0$, *there exists a region* $\tilde{\mathcal{N}}_\eta \subset \eta$ *such that the atoms* $F_{\lambda'}$, $\lambda' = (\eta', \theta', \beta)$ *overlapping* $\tilde{\mathcal{N}}_\eta$ *are linearly independent over* $\tilde{\mathcal{N}}_\eta$, *i.e.*

$$\sum_{\lambda'\in\Lambda_m:\eta'\cap\mathcal{N}'_\eta\neq\emptyset} c_{\lambda'} F_{\lambda'}(x) = 0,\ x \in \tilde{\mathcal{N}}_\eta,\ \Longrightarrow\ c_{\lambda'} = 0,\ \lambda' \in \Lambda_m : \eta' \cap \tilde{\mathcal{N}}_\eta \neq \emptyset. \tag{3.42}$$

Let us verify the validity of (LLIN′) for two scenarios already discussed in Sect. 3.1.

Sparsely shifted B-splines: We adhere to the setting described earlier in Sect. 3.1 and consider tensor product B-splines of coordinate degree K and maximal smoothness $K - 1$ shifted on a regular grid in such a way that polynomial regions match for overlapping supports and that the resulting cover is sparse in the sense of Remark 2.2, see also the example following Remark 2.1. As explained before, for instance, for cardinal B-splines the supports are shifts of $[0, K+1]^d$ and we could shift on the lattice $L\mathbb{Z}^d$ for some $L \in \mathbb{N}$, $L \leq K$, e.g., $L = K$ (but close to K to have only a fixed number of overlaps independent of K). To see that Property (LLIN′) holds, it suffices to consider the coarsest level. Then the support Ω_θ of each ϕ_θ is a cube of side length $L - 1 = K - 1$. Therefore, when creating higher levels by dyadic subdivisions of the ground lattice, each φ_η for $\eta \in \Theta_{m+1}$ at the next higher level has the property that Ω_η has a nonzero intersection with an Ω_θ for only one $\theta \in \Theta_m$. Since on $\tilde{\mathcal{N}}_\eta := \Omega_\eta \cap \Omega_\theta$ the bumps φ_θ and φ_η are constant the validity of Property (LLIN′) is immediate.

This can be extended somewhat to more general sparse covers, where the validity of Property (LLIN′) can be reduced to a property of the basic atoms.

Proposition 3.8 *Property (LLIN′) holds for a sparse cover if the following is true: For each $\eta \in \Theta_{m+1}$ there exists a neighborhood $\tilde{\mathcal{N}}_\eta \subseteq \Omega_\eta$ such that*

$$\sum_{\substack{\theta \in \Theta_m,\ \theta \cap \mathcal{N}_\eta \neq \emptyset \\ |\beta| < k}} c_{\beta,\theta} P_{\eta,\beta} \phi_\theta(x) = 0, \quad x \in \tilde{\mathcal{N}}_\eta \implies c_{\beta,\theta} = 0,\ \theta \in \Theta_m,$$
$$\theta \cap \mathcal{N}_\eta \neq \emptyset,\ |\beta| < k, \tag{3.43}$$

where $\phi_\theta := \phi \circ A_\theta^{-1}$.

Proof Suppose that, in view of (2.5), (2.6), B_η is again a ball in $\eta \in \Theta_{m+1}$ which is not intersected by any other $\eta' \in \Theta_{m+1}$. Then, since B_η is overlapped only by η itself and since $\varphi_\eta \equiv 1$ on B_η we have

$$\sum_{\lambda' \in \Gamma_\eta^{m,m+1}} c_{\lambda'} F_{\lambda'}(x) = 0 \text{ on } B_\eta \iff \sum_{|\beta'|<k} \sum_{\theta' \cap B_\eta \neq \emptyset} c_{\eta,\beta',\theta'} P_{\eta,\beta'}(x) \varphi_{\theta'}(x) = 0 \text{ on } B_\eta.$$

Since the φ_θ and ϕ_θ differ only by one common factor we see that the $F_{\lambda'}$ that overlap B_η are linearly independent on B_η. Thus, Property (LLIN′) is valid. □

The second scenario concerns the
Variable order radial polynomial bumps: Again we have to extend formally the setting described in Property (LLIN′) to admit bumps ϕ^ν where pairwise different colors appear in the linear combination of product atoms in Property (LLIN′). The required local linear independence is then again an immediate consequence of the sufficiently large degree differences between different colors that has been chosen to account for the possible product combinations appearing in the F_λ.

Theorem 3.9 *Suppose that in addition to the assumptions in Theorem 3.3 Property (LLIN′) is valid. Then each collection*

$$\{F_\lambda : \lambda \in \Lambda_m\}, \quad m = 0, 1, \ldots,$$

is linearly independent on Ω and hence forms a basis for

$$W_m := \operatorname{span}\{F_\lambda : \lambda \in \Lambda_m).$$

Moreover, any $g \in W_m$ has a unique representation

$$g = \sum_{\lambda \in \Lambda_m} c_\lambda(g) F_\lambda, \tag{3.44}$$

where as in (3.13) the dual functionals c_λ, $\lambda = (\eta, \theta, \beta)$, have a representation $c_\lambda(g) = \langle g, c_\lambda \rangle_{B_\eta}$, for some $B_\eta \subset \eta$ which is comparable in size to η. Hence the functionals $c_\lambda(\cdot)$ are bounded linear functionals on $L_2(\Omega)$ and satisfy

$$|c_\lambda(g)| \lesssim |\eta|^{-1/2} \|g\|_{L_2(\eta)}, \quad \lambda = (\eta, \theta, \beta) \quad \forall g \in W_m, \tag{3.45}$$

where the constant depends only on k, $\mathbf{p}(\Theta)$ and ϕ.

Proof Under the given assumptions the construction of the dual functionals is analogous to the one given in the proof of Theorem 3.3. By an analogous dilation argument as in Lemma 3.4 one can establish again the fact that for some constant $b_5 > 0$ and a suitable $B_\eta \subset \eta$ one has

$$B_\eta \cap \eta' \cap \theta \neq \emptyset \implies |B_\eta \cap \eta' \cap \theta| \geq b_5 |\eta|. \tag{3.46}$$

Since the remaining assertions are analogous consequences the proof is complete. □

An immediate consequence of Theorem 3.9 can be stated as follows (see also (3.15)).

Corollary 3.10 *For any $g \in W_m$ we have*

$$\|g\|_2 \sim \left(\sum_{\lambda \in \Lambda_m} \|c_\lambda(g) F_\lambda\|_2^2 \right)^{1/2}. \tag{3.47}$$

In the following we shall frequently use the following relation

$$\|F_\lambda\|_2 \sim |\eta_\lambda|^{\frac{1}{2}} \|F_\lambda\|_\infty \tag{3.48}$$

which holds with constants depending on the polynomial degrees of the atoms.

4 Application to preconditioning for elliptic boundary value problems

We now turn to discretizations by means of the above type of partition of unity hierarchies. Thus, for any given $f \in V'$, V a Hilbert space and $a(\cdot, \cdot)$ a symmetric V-elliptic bilinear form (see (1.1)) we consider the problem: Find $u \in V$ such that

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V. \tag{4.1}$$

For simplicity we confine the discussion to the model case $V = H_0^1(\Omega)$. Higher order problems could be treated in an analogous way. The homogeneous boundary conditions are always supposed to be realized in the trial spaces by suitable polynomial factors in the atoms.

Since we shall not deal with discretizations for a fixed level m of resolution but wish to incorporate from the beginning the realization of adaptivity admissible trial functions should in principle, be atoms from *all* levels. More precisely, we shall make use of the atoms F_λ, defined in (3.37) for $\lambda \in \Lambda = \bigcup_{m=-1}^{\infty} \Lambda_m$, see (3.36).

We shall place this in the context of *stable splittings* in the theory of *multilevel Schwarz preconditioners* developed by many researchers, see e.g., [14,16] and the literature cited there. Here we adhere mainly to the findings in [9,16]. To this end, let $V_\lambda := \text{span}\,(F_\lambda)$ (see (3.37)) so that $H_0^1(\Omega) := V = \sum_\lambda V_\lambda$. The following is the main result of this section whose proof will be postponed.

Theorem 4.1 *The* $\{V_\lambda\}_{\lambda\in\Lambda}$ *form a* stable splitting *for* V *in the sense that there exist positive finite constants* c_V, C_V*, depending only on* $\mathbf{p}(\Theta)$, k *and* ϕ*, such that*

$$c_V \|v\|_V \le \inf_{v=\sum_\lambda v_\lambda} \left(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-2/d} \|v_\lambda\|_2^2 \right)^{1/2} \le C_V \|v\|_V. \tag{4.2}$$

Moreover, defining $\Lambda^m := \bigcup_{j=-1}^{m} \Lambda_j$*, the* $\{V_\lambda\}_{\lambda\in\Lambda^m}$ *form a uniformly stable splitting for the spaces* S_m *in the sense of* (4.2) *with the same constants* c_V, C_V*.*

This allows us to invoke the theory of Schwarz methods along the following lines. For $V_0 := S_0 = \text{span}\,\Phi_0$ define $P_{V_0} : V \to V_0$ and $r_{V_0} \in S_0$ by

$$a(P_{V_0} v, F_\lambda) = a(v, F_\lambda), \quad (r_{V_0}, F_\lambda)_{L_2} = \langle f, F_\lambda\rangle, \quad \lambda \in \Lambda_0 = \Theta_0.$$

Moreover, introducing the auxiliary bilinear forms:

$$b_\lambda(v, w) := |\eta_\lambda|^{-2/d} (v, w)_{L_2}, \quad v, w \in V_\lambda, \quad \lambda \in \Lambda \backslash \Lambda_0, \tag{4.3}$$

we endow the spaces V_λ with the norms $\|v\|_{V_\lambda} := (b_\lambda(v, v))^{1/2}$ and define the linear operator $P_{V_\lambda} : V \to V_\lambda$ and $f_\lambda \in V_\lambda$ by

$$\begin{aligned} |\eta_\lambda|^{-2/d} (P_{V_\lambda} v, F_\lambda)_{L_2} &= a(v, F_\lambda), \\ |\eta_\lambda|^{-2/d} (f_\lambda, F_\lambda)_{L_2} &= \langle f, F_\lambda\rangle. \end{aligned} \tag{4.4}$$

Thus, as usual,

$$P_{V_\lambda} v = a_\lambda(v) F_\lambda, \quad f_\lambda = r_\lambda(f) F_\lambda, \tag{4.5}$$

with

$$a_\lambda(v) = \frac{|\eta_\lambda|^{2/d} a(v, F_\lambda)}{\langle F_\lambda, F_\lambda\rangle}, \quad r_\lambda(f) = \frac{|\eta_\lambda|^{2/d} \langle f, F_\lambda\rangle}{\langle F_\lambda, F_\lambda\rangle}. \tag{4.6}$$

The following statements are now an immediate consequence of the results in [9,16].

Theorem 4.2 *Problem* (4.1) *is equivalent to the operator equation*

$$P_V u = \bar{f}, \tag{4.7}$$

where

$$P_V := P_{V_0} + \sum_{\lambda\in\Lambda\setminus\Lambda_0} P_{V_\lambda}, \quad \bar{f} := r_{V_0} + \sum_{\lambda\in\Lambda\setminus\Lambda_0} f_\lambda. \tag{4.8}$$

Moreover, the spectral condition number $\kappa(P_V)$ of the additive Schwarz operator P_V satisfies

$$\kappa(P_V) \le \frac{C_a C_V}{c_a c_V}, \tag{4.9}$$

where c_a, C_a, c_V, C_V are the constants from (1.1) *and* (4.2).

This latter fact implies that simple iterative schemes, such as Richardson iterations,

$$u^{n+1} = u^n + \alpha(\bar{f} - P_V u^n), \quad n = 0, 1, 2, \dots, \tag{4.10}$$

converge with a fixed error reduction rate per step. More specifically, suppose that $u^n = \sum_{\lambda\in\Lambda} u_\lambda^n F_\lambda$ with coefficient array $\mathbf{u}^n = (u_\lambda^n)_{\lambda\in\Lambda}$, (4.10) can be rephrased, in view of (4.5), (4.6) as

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \alpha(\bar{\mathbf{f}} - \mathbb{A}\mathbf{u}^n), \quad \mathbb{A}_{\lambda,\lambda'} = |\eta_\lambda|^{2/d} \|F_\lambda\|_2^{-2} a(F_\lambda, F_{\lambda'}), \quad \lambda, \lambda' \in \Lambda. \tag{4.11}$$

A few comments are in order. First of all, the above operator equation (4.7) is formulated in the full infinite dimensional space. Alternatively, restricting the summation to any a priori chosen finite subset $\bar{\Lambda}$ of Λ (e.g., $\bar{\Lambda} = \Lambda^m$), we obtain a finite dimensional discrete problem whose condition fulfills, in view of the second part of Theorem 4.1, the same bound, uniformly in the size and choice of $\bar{\Lambda}$. In this sense we have an asymptotically optimal preconditioner.

On the other hand, it is conceptually useful to consider the full infinite dimensional problem (4.7). In this case (4.10) is to be understood as an *idealized* scheme whose numerical implementation requires appropriate *approximate* applications of the (infinite dimensional) operator P_V quite in the spirit of [2]. This can be done by computing in addition to solving the coarse scale problem on $S_0 = V_0$ only finitely many but properly selected components P_{V_λ} each requiring only the solution of a one-dimensional problem. This hints at the adaptive potential of such an approach similar to the developments in [2]. Roughly speaking, one could try to monitor the size of the components of the weighted residual $\alpha(\bar{f} - P_V u^n)$ so as to replace it within a suitable tolerance by a vector of possibly small support. Thereby one would try to keep the supports of the approximations $\mathbf{u}^n$ as small as possible again within a desired gain of accuracy. This, in turn, raises the question which accuracy can be achieved at best when using linear combinations of at most N of the atoms, i.e. we are interested in the error of *best N-term approximation*

$$\sigma_{N,X}(v) := \inf \left\{ \left\| v - \sum_{\lambda\in\tilde{\Lambda}} a_\lambda F_\lambda \right\|_X : a_\lambda \in \mathbb{R},\ \#\tilde{\Lambda} \le N \right\}. \tag{4.12}$$

To see whether any adaptive strategy could offer a gain over simple uniform refinements it is interesting to understand the corresponding *approximation spaces*

$$\mathcal{A}^s_X := \left\{ v \in V : |v|_{\mathcal{A}^s_X} := \sup_{N\in\mathbb{N}} N^s \sigma_{N,X}(v) < \infty \right\}. \tag{4.13}$$

It is shown in [3] that, for instance, the Besov spaces $B_q^{1+ds}(L_p(\Omega))$ are embedded in $\mathcal{A}^s_{H^1}$, i.e. one has

$$\sigma_{N,H^1}(v) \lesssim N^{-\alpha/d} |v|_{B_q^{1+\alpha}(L_p(\Omega))},$$

provided that

$$\frac{1}{p} < \frac{\alpha}{d} + \frac{1}{2},$$

(see below for the definition of these spaces). Thus, the smaller p the more singular behavior of a solution can be compensated by nonlinear approximation so as to maintain the approximation order $N^{-\alpha/d}$. To achieve the same order through uniform discretizations the solution would have to belong to $B_\infty^{1+\alpha}(L_2(\Omega))$ which is a much smaller space (very close to $H^{1+\alpha}(\Omega)$).

A more thorough discussion of related adaptive solution schemes will be given elsewhere. The remainder of this note is devoted to the proof of the above stable splittings.

5 Besov spaces and stable splittings

For variational problems of the type considered in the previous section the energy space V is typically a Sobolev space. A common strategy for establishing the stability (4.2) of the splitting $\{F_\lambda\}_{\lambda\in\Lambda}$ required in Theorem 4.2 in this context is to exploit that the Sobolev spaces $H^t(\Omega)$ (or corresponding subspaces with vanishing traces) agree with the Besov spaces $B_2^t(L_2(\Omega))$ with equivalent norms and that the Besov norms are more suitable for analyzing multilevel splittings. Moreover, Besov spaces on $L_p(\Omega)$ for $p \neq 2$ are relevant for the analysis of nonlinear approximation such as best N-term approximation. Let us briefly recall that the Besov space $B_q^\alpha(L_p(\Omega))$, with $\alpha > 0$ and

$0 < p, q \le \infty$, is usually defined as the set of all functions $f \in L_p(\Omega)$ such that

$$|f|_{B_q^\alpha(L_p(\Omega))} := \left(\int_0^\infty (t^{-\alpha} \omega_k(f,t)_p)^q \frac{dt}{t} \right)^{1/q} < \infty \tag{5.1}$$

with the usual modification when $q = \infty$. As before $\omega_k(f,t)_p$ is the kth modulus of smoothness of f in L_p over Ω. The norm in $B_q^\alpha(L_p(\Omega))$ is defined by

$$\|f\|_{B_q^\alpha(L_p(\Omega))} := |\Omega|^{-\alpha/d} \|f\|_{L_p} + |f|_{B_q^\alpha(L_p(\Omega))}.$$

It is not hard to see that

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \left(\sum_{j=0}^\infty \left(2^{\alpha j} \omega_k(f, 2^{-j})_p \right)^{q/p} \right)^{1/q} \tag{5.2}$$

Moreover, following [3,12], the moduli of smoothness can be localized which allows us to related the Besov norms to the cover Θ from Sect. 2 by verifying that

$$|f|_{B_q^\alpha(L_p(\Omega))} \sim \left(\sum_{m=0}^\infty \left(\sum_{\theta \in \Theta_m} |\theta|^{-\alpha p/d} \omega_k(f,\theta)_p^p \right)^{q/p} \right)^{1/q}. \tag{5.3}$$

To see how this, in turn, can be related to norms of the type appearing in (4.2), it will be convenient to introduce next a scale of "smoothness spaces" (B-spaces) induced by multilevel covers Θ as described in Sect. 2. The construction of these spaces is inspired by previous work referring to a different setting, see [4,12,15]. As before we assume that Ω is a bounded extension domain in $\mathbb{R}^d$ as explained in Sect. 1. Since in the context of Theorem 4.1 we are interested here in characterizing only the Sobolev spaces $H^t = B_2^t(L_2)$, we shall confine the subsequent discussion to the case $p = q = 2$ and refer to [3] for the general case.

The following first version defines the B-space $\mathcal{B}^s(\Theta)$ via atomic decompositions which will provide our link to the stable splittings in Theorem 4.1. More precisely, the B-space $\mathcal{B}^s(\Theta)$, $s > 0$, is defined as the set of all functions $f \in L_2(\Omega)$ such that

$$\|f\|_{\mathcal{B}^s(\Theta)} := \inf_{f = \sum_{\lambda \in \Lambda} a_\lambda F_\lambda} \left(\sum_{\lambda \in \Lambda} |\theta_\lambda|^{-2s} \|a_\lambda F_\lambda\|_2^2 \right)^{1/2} < \infty, \tag{5.4}$$

where the infimum is taken over all representations $f = \sum_{\lambda \in \Lambda} a_\lambda F_\lambda$ in $L_2(\Omega)$. Here $\Lambda := \cup_{m=-1}^\infty \Lambda_m$, $\Lambda_{-1} := \Theta_0$.

A second approach to the B-spaces $\mathcal{B}^s(\Theta)$, that will help us to interrelate the above norms, is through quasi-interpolants. For $f \in L_2(\Omega)$ we have by Theorem 3.7

$$f = Q_0 f + \sum_{m=0}^{\infty} (Q_{m+1} f - Q_m f) = \sum_{m=-1}^{\infty} \sum_{\lambda \in \Lambda_m} d_\lambda(f) F_\lambda. \tag{5.5}$$

We define

$$\|f\|^{Q}_{\mathcal{B}^s(\Theta)} := \left(\sum_{\lambda \in \Lambda} |\theta_\lambda|^{-2s} \|d_\lambda(f) F_\lambda\|_2^2 \right)^{1/2}, \tag{5.6}$$

where $\{d_\lambda(f)\}_{\lambda \in \Lambda}$ comes from (5.5).

These B-spaces are conveniently linked to Besov spaces by introducing the third version through the semi-norm

$$|f|^{\omega}_{\mathcal{B}^s(\Theta)} := \left(\sum_{\theta \in \Theta} |\theta|^{-2s} \omega_k(f, \theta)_2^2 \right)^{1/2} < \infty, \tag{5.7}$$

where $\omega_k(f, \theta)_2$ is again the kth modulus of smoothness of f on θ in L_2. We set

$$\|f\|^{\omega}_{\mathcal{B}^s(\Theta)} := |\Omega|^{-s} \|f\|_2 + |f|^{\omega}_{\mathcal{B}^s(\Theta)}. \tag{5.8}$$

Evidently, $\|\cdot\|^{\omega}_{\mathcal{B}^s(\Theta)}$ is a norm. This norm now depends on one more parameter $k \geq 1$ which we shall not indicate explicitly in the notation before we clearly exhibit its role. We shall assume at this point, however, that $k \leq r$, where $r > 0$ is the smoothness of our building blocks ϕ (see the beginning of Sect. 3).

A glance at (5.3) reveals that the latter norm is just the Besov norm with $p = q = 2$ where the smoothness index is rescaled, i.e. s plays the role of α/d.

Remark 5.1 For $0 < s < k/d$, we have $\mathcal{B}^s(\Theta) = B_2^{ds}(L_2(\Omega))$ and for f in this space one has

$$\|f\|_{\mathcal{B}^s(\Theta)} \sim \|f\|_{B_2^{ds}(L_2(\Omega))} \sim \|f\|_{H^{ds}(\Omega)}.$$

Without further mentioning we assume in the following that Property (LLIN′) or the hypotheses of Proposition 3.8 are valid.

The main result of this section concerns the following interrelation of the above norms.

Theorem 5.2 *Let $s > 0$, and $k \geq 1$.*

(a) *If $f \in \mathcal{B}^s(\Theta)$, then*

$$\|f\|_{\mathcal{B}^s(\Theta)} \leq \|f\|^{Q}_{\mathcal{B}^s(\Theta)} \lesssim \|f\|^{\omega}_{\mathcal{B}^s(\Theta)}. \tag{5.9}$$

(b) *The norms* $\|\cdot\|_{\mathcal{B}^s(\Theta)}$, $\|\cdot\|^{Q}_{\mathcal{B}^s(\Theta)}$, *and* $\|\cdot\|^{\omega}_{\mathcal{B}^s(\Theta)}$, *defined in* (5.4),(5.6) *and* (5.8), *are equivalent for* $0 < s < k/d$. *Here the constants depend only on* s, k, *on the parameters in* $\mathbf{p}(\Theta)$ *of* Θ, *and on* ϕ.

Proof As for (a), in view of the special decomposition $f = \sum_m (Q_m - Q_{m-1}) f$, the first inequality is trivial. To confirm the second inequality, we recall that, by (3.47)

$$\begin{aligned}\sum_{\lambda\in\Lambda_m} \|d_\lambda(f) F_\lambda\|_2^2 &\sim \|(Q_{m+1} - Q_m) f\|_2^2 \le \sum_{\theta\in\Theta_m} \|(Q_{m+1} - Q_m) f\|^2_{L_2(\theta)} \\ &\lesssim \sum_{\theta\in\Theta_m} \omega_k(f,\theta)_2^2,\end{aligned}$$

where we have used in the last step (3.31), (C2), (C3), (C5) as well as standard properties of the modulus of smoothness. The right-hand-side inequality in (5.9) is now an immediate consequence of definition (5.7).

To confirm (b) it remains to show that

$$\|f\|^{\omega}_{\mathcal{B}^s(\Theta)} \lesssim \|f\|_{\mathcal{B}^s(\Theta)}. \tag{5.10}$$

To this end, we need to estimate $\omega_k(f,\theta)_2^2$ which requires the following simple technical observations. Recalling that by the properties (C1)–(C5) our normalization ensures that $\|F_\lambda\|_\infty \sim 1$, one derives that

$$\left\|\partial^\alpha F_\lambda\right\|_\infty \lesssim |\eta_\lambda|^{-|\alpha|/d}, \quad |\alpha| \le k.$$

Hence for some $h \in \mathbb{R}^d$, $|h| \le \operatorname{diam}\theta \sim |\theta|^{1/d}$, denoting by $\partial f/\partial h = \lim_{t\to 0}(f(\cdot + th/|h|) - f(\cdot))/t$ the directional derivative (see p1, p2, (C1))

$$\begin{aligned}\omega_k(F_\lambda,\theta)_2^2 &\lesssim |h|^{2k} \left\|\left(\frac{\partial}{\partial h}\right)^k F_\lambda\right\|_\infty^2 |\theta| \le |\theta|^{2k/d} |\eta_\lambda|^{-2k/d} |\theta| \\ &= \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2k/d} |\theta| \lesssim \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{2k}{d}+1} \|F_\lambda\|_2^2,\end{aligned} \tag{5.11}$$

where we used that $\|F_\lambda\|_2 \sim |\eta_\lambda|^{1/2}$ (see (3.48)) due to the normalization $\|F_\lambda\|_\infty \sim 1$.

To prove now (5.10) consider any decomposition $f = \sum_{\lambda\in\Lambda} a_\lambda F_\lambda$ in L_2. Employing (5.11), we have for $\theta \in \Theta$

$$\begin{aligned}\omega_k(f,\theta)_2 &\lesssim \omega_k\left(\sum_{|\eta_\lambda|>|\theta|} a_\lambda F_\lambda, \theta\right)_2 + \left\|\sum_{|\eta_\lambda|\le|\theta|} a_\lambda F_\lambda\right\|_{L_2(\theta^*)} \\ &\lesssim \sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset} |a_\lambda|\omega_k(F_\lambda,\theta)_2 + \left\|\sum_{|\eta_\lambda|\le|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda\right\|_2 \qquad (5.12)\\ &\lesssim \sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{k}{d}+\frac{1}{2}} \|a_\lambda F_\lambda\|_2 + \left\|\sum_{|\eta_\lambda|\le|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda\right\|_2.\end{aligned}$$

Now by (5.7) and (5.12), we infer

$$\begin{aligned}\left(|f|^{\omega}_{\mathcal{B}^s(\Theta)}\right)^2 &= \sum_{\theta\in\Theta} |\theta|^{-2s}\omega_k(f,\theta)_2^2 \\ &\lesssim \sum_{\theta\in\Theta}|\theta|^{-2s}\left[\sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset}\left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{k}{d}+\frac{1}{2}}\|a_\lambda F_\lambda\|_2\right]^2 \\ &\quad + \sum_{\theta\in\Theta}|\theta|^{-2s}\left\|\sum_{|\eta_\lambda|\le|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda\right\|_2^2 \\ &=: \Sigma_1 + \Sigma_2. \qquad (5.13)\end{aligned}$$

For the first sum, we have

$$\begin{aligned}\Sigma_1 &= \sum_{\theta\in\Theta}\left[\sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset}\left(\frac{|\theta|}{|\eta_\lambda|}\right)^{\frac{k}{d}-s+\frac{1}{2}}|\eta_\lambda|^{-s}\|a_\lambda F_\lambda\|_2\right]^2 \\ &= \sum_{\theta\in\Theta}\left[\sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset}\left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta+\frac{1}{2}} A_\lambda\right]^2, \qquad (5.14)\end{aligned}$$

where $2\delta := k/d - s > 0$ and $A_\lambda := |\eta_\lambda|^{-s}\|a_\lambda F_\lambda\|_2$. Applying Cauchy–Schwarz's inequality, we get

$$\Sigma_1 \le \sum_{\theta\in\Theta}\left[\sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset}\left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta}\right]\sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset}\left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta+1} A_\lambda^2. \qquad (5.15)$$

Similarly as above for $\theta \in \Theta_m$ and $\eta_\lambda \in \Theta_{m-\nu}$ one has $|\theta|/|\eta_\lambda| \lesssim 2^{-\nu a_0}$. Consequently,

$$\sum_{|\eta_\lambda|>|\theta|,\ \eta_\lambda\cap\theta\neq\emptyset}\left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta} \lesssim \sum_{\nu=0}^{\infty} 2^{-\nu a_0 2\delta} \lesssim 1. \qquad (5.16)$$

We use this in (5.15) and switch the order of summation to obtain

$$\Sigma_1 \lesssim \sum_{\lambda\in\Lambda} A_\lambda^2 \sum_{|\theta|<|\eta_\lambda|,\,\theta\cap\eta_\lambda\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta+1}. \tag{5.17}$$

Fix $\lambda \in \Lambda$ and assume that $\eta_\lambda \in \Theta_j$. From properties (C1) and (C5) we infer that the number of cells $\theta \in \Theta_{j+l}$ whose support intersect η_λ is bounded by $c2^{la_0}$ to obtain

$$\sum_{|\theta|<|\eta_\lambda|,\,\theta\cap\eta_\lambda\neq\emptyset} \left(\frac{|\theta|}{|\eta_\lambda|}\right)^{2\delta+1} \lesssim \sum_{l=0}^{\infty} \sum_{\theta\in\Theta_{j+l},\,\theta\cap\eta_\lambda\neq\emptyset} 2^{-la_0(1+2\delta)} \lesssim \sum_{l=0}^{\infty} 2^{-la_0 2\delta} \lesssim 1.$$

Inserting this in (5.17) we get

$$\Sigma_1^{1/2} \lesssim \|f\|_{\mathcal{B}^s(\Theta)}. \tag{5.18}$$

We now estimate Σ_2. Note first that by (C2) it follows that

$$\Big\| \sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset} a_\lambda F_\lambda \Big\|_2^2 \lesssim \sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset} \|a_\lambda F_\lambda\|_2^2, \quad \text{if } \theta\in\Theta_m, \quad \nu\geq 0.$$

Hence

$$\begin{aligned}\Sigma_2 &\lesssim \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m} |\theta|^{-2s} \left[\sum_{\nu=0}^{\infty}\left(\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset} \|a_\lambda F_\lambda\|_2^2\right)^{1/2}\right]^2 \\ &= \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m} \left[\sum_{\nu=0}^{\infty}\left(\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\eta_\lambda|}{|\theta|}\right)^{2s} |\eta_\lambda|^{-2s}\|a_\lambda F_\lambda\|_2^2\right)^{1/2}\right]^2.\end{aligned}$$

As above we denote $A_\lambda := |\eta_\lambda|^{-s}\|a_\lambda F_\lambda\|_2$ and use that $|\eta_\lambda|/|\theta| \lesssim 2^{-\nu a_0}$ if $\theta \in \Theta_m$, $\eta_\lambda \in \Theta_{m+\nu}$ to obtain

$$\begin{aligned}\Sigma_2 &\lesssim \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m} \left[\sum_{\nu=0}^{\infty} 2^{-\nu a_0 s/2}\left(\sum_{\eta_\lambda\in\Theta_{m+\nu},\,\eta_\lambda\cap\theta\neq\emptyset} \left(\frac{|\eta_\lambda|}{|\theta|}\right)^{s} A_\lambda^2\right)^{1/2}\right]^2 \\ &=: \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m} \sigma_\theta.\end{aligned} \tag{5.19}$$

Now applying Cauchy–Schwarz's inequality, we have

$$\sigma_\theta \le \left(\sum_{\nu=0}^{\infty} 2^{-\nu a_0 s}\right) \sum_{\nu=0}^{\infty} \sum_{\eta_\lambda \in \Theta_{m+\nu},\, \eta_\lambda \cap \theta \neq \emptyset} \left(\frac{|\eta_\lambda|}{|\theta|}\right)^s A_\lambda^2$$
$$\lesssim \sum_{\nu=0}^{\infty} \sum_{\eta_\lambda \in \Theta_{m+\nu},\, \eta_\lambda \cap \theta \neq \emptyset} \left(\frac{|\eta_\lambda|}{|\theta|}\right)^s A_\lambda^2.$$

Substituting this in (5.19) and switching the order of summation, we obtain

$$\Sigma_2 \lesssim \sum_{\lambda \in \Lambda} A_\lambda^2 \sum_{|\theta| \ge |\eta_\lambda|,\, \theta \cap \eta_\lambda \neq \emptyset} \left(\frac{|\eta_\lambda|}{|\theta|}\right)^s .$$

Exactly as in (5.16) the second sum above can be bounded from above by a constant, which implies $\Sigma_2^{1/2} \lesssim \|f\|_{\mathcal{B}^s(\Theta)}$. This coupled with (5.18) yields $|f|^{\omega}_{\mathcal{B}^s(\Theta)} \lesssim \|f\|_{\mathcal{B}^s(\Theta)}$.

The estimate $|\Omega|^{-s}\|f\|_2 \lesssim \|f\|_{\mathcal{B}^s(\Theta)}$ is similar to the estimate of Σ_2 above but is easier and its proof will be omitted. The proof of (5.10) is complete. □

An immediate further consequence of (5.3) is the following fact.

Corollary 5.3 *Under the above assumptions the norms* $\|\cdot\|_{B_2^{sd}(L_2(\Omega))}$, $\|\cdot\|^{\omega}_{\mathcal{B}^s(\Theta)}$, $\|\cdot\|_{\mathcal{B}^s(\Theta)}$, *and* $\|\cdot\|^{Q}_{\mathcal{B}^s(\Theta)}$, *defined in* (5.1), (5.8), (5.4) *and* (5.6), *are equivalent for* $0 < s < k/d$.

Since the norms $a(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent, employing the well known fact that

$$\|\cdot\|_{H^1(\Omega)}, \sim \|\cdot\|_{B_2^1(L_2(\Omega))},$$

mentioned earlier, the first part of the assertion of Theorem 4.1 is immediate. The remaining part, stating that finite subsets of the splitting for all of V still form uniformly stable frames follows from the fact that the telescoping expansions underlying the version $\|\cdot\|^{Q}_{\mathcal{B}^s(\Theta)}$, terminate without affecting this norm. Thus the proof of Theorem 4.1 is complete.

We wish to conclude the discussion with some comments relating back to the notion of best N-term approximation addressed at the end of the previous section and may be viewed as another byproduct of the above norm equivalences. Of course, in the present context we are interested in $\sigma_{N,X}$ for $X = H_0^1(\Omega)$, the energy space of second order elliptic problems. To this end, consider in analogy to (5.6)

$$\|f\|^{Q}_{\mathcal{B}^s_p(\Theta)} := \left(\sum_{\lambda \in \Lambda} |\theta_\lambda|^{-sp} \|d_\lambda(f) F_\lambda\|_p^p\right)^{1/p}, \tag{5.20}$$

for any $0 < p \le \infty$. For $p < 1$ the coefficients $d_\lambda(f)$ need to be defined with the aid of somewhat different quasiinterpolants and the spaces $\mathcal{B}^s_p(\Theta)$ can again be shown to coincide within a certain range with the Besov spaces $B_p^{s/d}(L_p(\Omega))$, we refer to [3] for the details.

Theorem 5.4 *Suppose that for some* $\alpha > 0$ *(under the assumptions in Section 4)* $v \in \mathcal{B}_p^{\alpha+1/d}(\Theta)$ *with*

$$\frac{1}{p} = \alpha + \frac{1}{2}. \tag{5.21}$$

Then

$$\sigma_{N,H_0^1(\Omega)}(v) \lesssim \|v\|_{\mathcal{B}_p^{\alpha+1/d}(\Theta)} N^{-\alpha}, \tag{5.22}$$

with a constant depending only on d, $\mathbf{p}(\Theta)$, k *and* ϕ. *Thus, whenever* $\alpha + 1/d < k/d$, *the Besov regularity* $v \in B_p^{1+\alpha d}(L_p(\Omega))$ *ensures a best* N*-term error decay rate of* $N^{-\alpha}$.

Proof Rearrange the terms $\{\| |\eta_\lambda|^{-1/d} d_\lambda(v) F_\lambda \|_2\}$ in decreasing order according to their size

$$\left\| |\eta_{\lambda_1}|^{-1/d} d_{\lambda_1}(v) F_{\lambda_1} \right\|_2 \geq \left\| |\eta_{\lambda_2}|^{-1/d} d_{\lambda_2}(v) F_{\lambda_2} \right\|_2 \geq \cdots$$

and set $S_N := \sum_{j=1}^N d_{\lambda_j}(v) F_{\lambda_j}$. Then by Theorem 5.2 and Corollary 5.3 we obtain, on account of the well-known characterization $\mathcal{A}_{\ell_2}^\alpha$ by the weak space ℓ_p^w, $\frac{1}{p} = \alpha + \frac{1}{2}$ (see below for the definition of the norm),

$$\begin{aligned} \|v - S_N\|_{H^1} &\sim \left\| \sum_{j=N+1}^{\infty} d_{\lambda_j}(v) F_{\lambda_j} \right\|_{H^1} \lesssim \left(\sum_{j=N+1}^{\infty} |\eta_{\lambda_j}|^{-2/d} \|d_{\lambda_j}(v) F_{\lambda_j}\|_2^2 \right)^{1/2} \\ &\lesssim N^{-\alpha} \left\| \{ |\eta_\lambda|^{-1/d} \|d_\lambda(v) F_\lambda\|_2 \} \right\|_{\ell_p^w}, \end{aligned}$$

where for the decreasing rearrangement $(a_j^*)_{j \in \mathbb{N}}$ of the sequence $\mathbf{a} = (a_\lambda)_{\lambda \in \Lambda}$

$$\|\mathbf{a}\|_{\ell_p^w} := \sup_{n \in \mathbb{N}} n^{1/p} |a_n^*|.$$

Since $\|\mathbf{a}\|_{\ell_p^w} \lesssim \|\mathbf{a}\|_{\ell_p}$ we conclude that

$$\begin{aligned}\|v - S_N\|_{H^1} &\lesssim N^{-\alpha}\left(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-p/d} \|d_\lambda(v)F_\lambda\|_2^p\right)^{1/p} \\ &\sim N^{-\alpha}\left(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-\frac{p}{d}} |\eta_\lambda|^{\frac{p}{2}-1} \|d_\lambda(v)F_\lambda\|_p^p\right)^{1/p} \\ &\sim N^{-\alpha}\left(\sum_{\lambda\in\Lambda} |\eta_\lambda|^{-\frac{p}{d}-\alpha p} \|d_\lambda(v)F_\lambda\|_p^p\right)^{1/p} = N^{-\alpha}\|v\|_{\mathcal{B}_p^{\alpha+1/d}(\Theta)},\end{aligned}$$

where we have used (3.48) and (5.21). In view of Corollary 5.1, this completes the proof. □

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