5.12. Sobolev Spaces

In the classical case of $M = \mathbb{R}^d$, the Sobolev spaces W_p^r, $r \in \mathbb{N}$, $1 \le p \le \infty$, are defined as the space of functions $f \in L^p$, whose distributional derivatives $\partial^\alpha f$ are also in L^p for any $\alpha \in \mathbb{Z}_+^d$, $|\alpha| \le r$. The norm of the space is then given by

$$\|f\|_{W_p^r(\mathbb{R}^d)} := \sum_{|\alpha| \le r} \|\partial^\alpha f\|_p. \tag{5.47}$$

It is known [**32**, §1.3], that for $1 < p < \infty$, an equivalent definition exists using the Fourier transform. In more generality, for any $s \in \mathbb{R}$, one may define the (inhomogeneous) Sobolev space $W_p^s(\mathbb{R}^d)$ as the space of all tempered distributions f for which

$$((1 + |\cdot|^2)^{s/2} \hat{f})^\vee \in L^p,$$

where $\hat{f}$ is the Fourier transform of f and $g^\vee$ is the inverse Fourier transform of g.

The norm is then defined by

$$\|f\|_{W_p^s(\mathbb{R}^d)} := \|((1+|\cdot|^2)^{s/2}\hat{f})^\vee\|_p. \tag{5.48}$$

The norms (5.47) and (5.48) are equivalent for the case where s is a non-negative integer and $1 < p < \infty$, while for a non-integer $s > 0$, we get the so called **Fractional Sobolev Spaces**.

Recall that for $M = \mathbb{R}^d$ and $L = -\Delta$, we have for $f \in D(L)$

$$Lf = (|\cdot|^2\hat{f})^\vee.$$

In correlation with (5.48), this leads to the following definition of Sobolev spaces in our setting

Definition 5.38. Let L be a non-negative self adjoint operator on M and let $s \in \mathbb{R}$. If for a measurable function f, $(I+L)^{s/2}f \in L^p$, $1 \le p \le \infty$, we say that $f \in W_p^s = W_p^s(L)$ and we define

$$\|f\|_{W_p^s(L)} := \|(I+L)^{s/2}f\|_p.$$

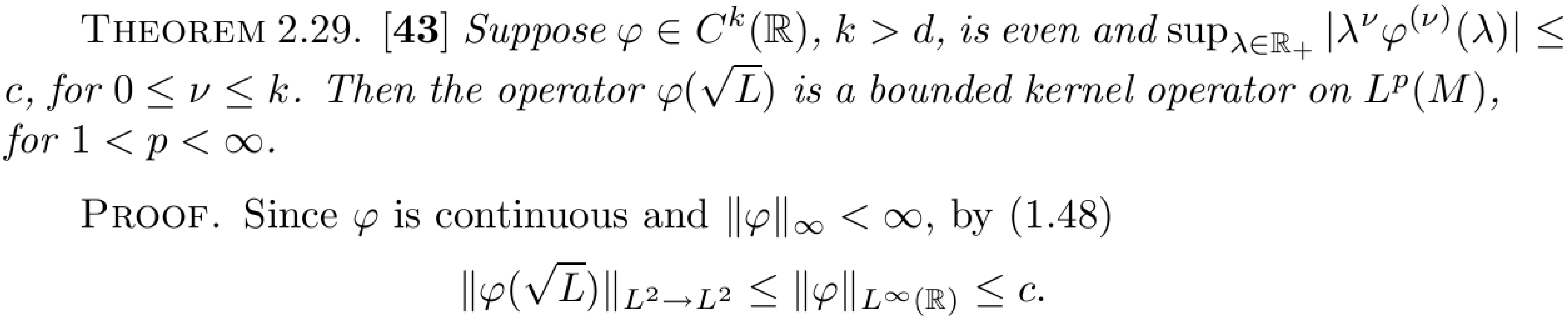
For the case $p = 2$, we get from (1.46)

$$\|f\|_{W_2^s(L)}^2 = \int_0^\infty (1+\lambda)^s d\|E_\lambda f\|_2^2.$$

First we would like to see the following

PROPOSITION 5.39. For $s > 0$ and $1 \le p \le \infty$, we have a continuous embedding $W_p^s \subset L^p$.

$$\boxed{\|\varphi(\sqrt{L})f\|_2^2 = \int_0^\infty |\varphi(\sqrt{\lambda})|^2 d\|E_\lambda f\|_2^2}$$



THEOREM 2.29. [**43**] *Suppose* $\varphi \in C^k(\mathbb{R})$, $k > d$, *is even and* $\sup_{\lambda\in\mathbb{R}_+} |\lambda^\nu \varphi^{(\nu)}(\lambda)| \leq c$, *for* $0 \leq \nu \leq k$. *Then the operator* $\varphi(\sqrt{L})$ *is a bounded kernel operator on* $L^p(M)$, *for* $1 < p < \infty$.

PROOF. Since φ is continuous and $\|\varphi\|_\infty < \infty$, by (1.48)

$$\|\varphi(\sqrt{L})\|_{L^2\to L^2} \leq \|\varphi\|_{L^\infty(\mathbb{R})} \leq c.$$

$$\|\varphi(\sqrt{L})f\|_2^2 = \int_0^\infty |\varphi(\sqrt{\lambda})|^2 d\|E_\lambda f\|_2^2 \leq \|\varphi\|_\infty^2 \|f\|_2^2.$$

We now show that the equivalence of the two Sobolev norms (5.47) and (5.48) for $M = \mathbb{R}^d$ can be generalized

PROPOSITION 5.40. Let $m \in \mathbb{N}$ and $1 < p < \infty$. Then $f \in W_p^{2m}$ if and only if $f \in L^p$ and $L^m f \in L^p$ and

$$\|(I+L)^m\|_p \sim \|f\|_p + \|L^m f\|_p.$$

PROOF. Assume first that $f \in L^p$ and $L^m f \in L^p$. Set

$$\theta(\lambda) := \frac{(1+\lambda^2)^m}{1+\lambda^{2m}}, \quad \lambda \in \mathbb{R}.$$

Clearly, $(I+L)^m = \theta(\sqrt{L})(I+L^m)$. Simple calculations show that

$$\sup_{\lambda \in \mathbb{R}} |\lambda^\nu \theta^{(\nu)}(\lambda)| < \infty, \quad \forall \nu \geq 0.$$

Since θ is also in $C^\infty(\mathbb{R})$ and is even, Theorem 2.29 implies that $\theta(\sqrt{L})$ is a bounded operator on L^p, $1 < p < \infty$. Therefore

$$\|(I+L)^m f\|_p \le \|\theta(\sqrt{L})\|_{L^p \to L^p} \|(I+L^m)f\|_p \le c(\|f\|_p + \|L^m f\|_p),$$

which yields $f \in W_p^{2m}$.

We now assume $f \in W_p^{2m}$ and prove the inverse embedding. Using Proposition 5.39 we already know that $f \in L^p$, and $\|f\|_p \le c\|f\|_{W_p^{2m}}$. It remains to show $\|L^m f\|_p \le c\|f\|_{W_p^{2m}}$. To this end we use

$$\tilde{\theta}(\lambda) := \frac{\lambda^{2m}}{(1+\lambda^2)^m}, \quad \lambda \in \mathbb{R},$$

to obtain the representation $L^m = \tilde{\theta}(\sqrt{L})(I+L)^m$ and proceed similarly as above.

□

5.2. Besov Spaces

We will introduce two versions of Besov spaces associated with L. The first are classic Besov spaces $B^s_{p,q}$, which for $s > 0$ and $p \geq 1$ can be identified as approximation spaces of linear approximation in L^p from the spectral spaces Σ^p_λ (see Definition 3.2). The second type $\tilde{B}^s_{p,q}$, are related to nonlinear approximation for certain indices.

First, in-line with previous definitions of admissible cut-off functions, we provide another type of admissibility which is less restrictive

Definition 5.15. We shall say that a pair of real-valued even functions $\varphi_0, \varphi \in C_0^\infty(\mathbb{R})$ are **admissible of weak-type** if they satisfy the following conditions for $b > 1$:

$$\operatorname{supp}(\varphi_0) \subset [-b, b], \quad |\varphi_0(\lambda)| \geq c > 0, \text{ for } \lambda \in [-b^{3/4}, b^{3/4}],$$
$$\operatorname{supp}(\varphi) \subset [-b, -b^{-1}] \cup [b^{-1}, b], \quad |\varphi(\lambda)| \geq c > 0 \text{ for } \lambda \in [-b^{3/4}, -b^{-3/4}] \cup [b^{-3/4}, b^{3/4}].$$

DEFINITION 5.17. Let L be a non-negative self-adjoint operator on M, $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Let $\varphi_0, \varphi \in C_0^\infty(\mathbb{R})$ be any admissible pair in the sense of Definition 5.15 with $b > 1$. The Besov space $B_{p,q}^s$, $q < \infty$, is defined as the set of all $f \in \mathcal{D}'$, such that

$$\|f\|_{B_{p,q}^s} := \left(\sum_{j=0}^{\infty} (b^{sj} \|\varphi_j(\sqrt{L}) f\|_p)^q \right)^{1/q} < \infty. \tag{5.14}$$

With $d > 0$ from (1.15) playing the role of dimension, the Besov space $\tilde{B}_{p,q}^s$, $q < \infty$, is defined as the set of all $f \in \mathcal{D}'$, such that

$$\|f\|_{\tilde{B}_{p,q}^s} := \left(\sum_{j=0}^{\infty} (\||B(\cdot, b^{-j})|^{-s/d} \varphi_j(\sqrt{L}) f\|_p)^q \right)^{1/q} < \infty. \tag{5.15}$$

The above discrete q-norms are replaced by the sup-norm if $q = \infty$.

It is easy to see that the Besov spaces $B^s_{p,q}$, $\tilde{B}^s_{p,q}$ are (quasi-)Banach spaces. We point out that the normalization of the norms of $\tilde{B}^s_{p,q}$ by the parameter d is in-line with the classic theory of Besov spaces on $\mathbb{R}^d$. We see that in special cases, such as $M = \mathbb{R}^d$, where $|B(x,r)| \sim r^d$, $\forall x \in M$, $r > 0$, then $B^s_{p,q} \sim \tilde{B}^s_{p,q}$.

Next, as in the classic theory for $M = \mathbb{R}^d$ [**32**, §2.2], we need to ensure the consistency of the definition of the Besov spaces in the following sense

THEOREM 5.18. [**43**] *Each of the two types of Besov spaces is equivalent with equivalent norms for any choices of admissible pairs* φ_0, φ *of weak type and parameter* $b > 1$.

5.4. Characterization of Besov Spaces by Linear Approximation From Spaces of Finite Spectra

It is natural and classical to characterize the Besov spaces $B^s_{p,q}$ with $s > 0$ and $1 \leq p \leq \infty$, by means of linear approximation from 'polynomial' spaces [**21**]. Here we show a generalized version for Σ^p_λ, $\lambda \geq 1$. First, recall from (3.24) that for $f \in L^p(M)$, $1 \leq p < \infty$, or $f \in UCB$ for $p = \infty$

$$\mathcal{E}_\lambda(f)_p := \inf_{g \in \Sigma^p_\lambda} \|f - g\|_p, \quad \lambda \geq 1.$$

Definition 5.21. The space $A^s_{p,q}$, $s > 0$, $0 < q \le \infty$, is defined as the set of functions $f \in L^p$, for which the following norm is finite

$$\|f\|_{A^s_{p,q}} := \|f\|_p + |f|_{A^s_{p,q}} = \|f\|_p + \begin{cases} \left(\sum_{j\ge 0} (2^{sj}\mathcal{E}_{2^j}(f)_p)^q\right)^{1/q}, & 0 < q < \infty, \\ \sup_{j\ge 0} 2^{sj}\mathcal{E}_{2^j}(f)_p, & q = \infty. \end{cases}$$

Theorem 5.22. [**10**] *Let $s > 0$, $1 \le p \le \infty$, and $0 < q \le \infty$. Then $f \in B^s_{p,q}$ if and only if $f \in A^s_{p,q}$ and*

$$\|f\|_{B^s_{p,q}} \sim \|f\|_{A^s_{p,q}}.$$

Preparation: Hardy discrete inequality

For $s > 0$ and $1 \leq q < \infty$, define the sequence norm for $a = \{a_k\}_{k=0}^{\infty}$

$$\|a\|_{s,q} := \left(\sum_{k=0}^{\infty} 2^{ks} |a_k| \right)^{1/q} .$$

Then, if for two sequences $a = \{a_k\}_{k=0}^{\infty}$,$b = \{b_k\}_{k=0}^{\infty}$, there exists $c_0 > 0$ such that

$$|b_k| \leq c_0 \sum_{j=k}^{\infty} |a_j|, \quad \forall k \geq 0,$$

then

$$\|b\|_{s,q} \leq c c_0 \|a\|_{s,q}.$$

Proof Take $0 < \theta < s$ and let $1/q + 1/q' = 1$. Then

$$\begin{aligned}
|b_k| &\leq c_0 \sum_{j=k}^{\infty} [2^{j\theta} |a_j|] 2^{-j\theta} \\
&\leq c_0 \left(\sum_{j=k}^{\infty} [2^{j\theta} |a_j|]^q \right)^{1/q} \left(\sum_{j=k}^{\infty} 2^{-j\theta q'} \right)^{1/q'} \\
&\leq c c_0 2^{-k\theta} \left(\sum_{j=k}^{\infty} [2^{j\theta} |a_j|]^q \right)^{1/q} .
\end{aligned}$$

This gives

$$
\begin{aligned}
\|b\|_{s,q}^q &= \sum_{k=0}^{\infty} [2^{k(s-\theta)} 2^{k\theta} |b_k|]^q \\
&\leq c \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} [2^{k(s-\theta)} 2^{j\theta} |a_j|]^q \\
&= c \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} [2^{k(s-\theta)} 2^{j(\theta-s)}]^q [2^{js} |a_j|]^q \\
&= c \sum_{j=0}^{\infty} [2^{js} |a_j|]^q \left(\sum_{k=0}^{j} 2^{(k-j)(s-\theta)q} \right) \\
&\leq c \|a\|_{s,q}^q .
\end{aligned}
$$

PROOF. Let φ_0, φ be a pair of admissible cut-off functions, with $b = 2$, $\operatorname{supp}(\varphi_0) \subset [-2, 2]$, $\operatorname{supp}(\varphi) \subset [-2, -1/2] \cup [1/2, 2]$, $\varphi_j := \varphi(2^{-j}\cdot)$, $j \geq 1$, satisfying $\sum_{j=0}^{\infty} \varphi_j \equiv 1$ (see §4.2.1). Suppose $f \in B^s_{p,q}$. We first need to verify that $f \in L^p$. It is easy to see that $\|f\|_{B^s_{p,\infty}} \leq \|f\|_{B^s_{p,q}}$, for any $0 < q \leq \infty$. This implies $\|\varphi_j(\sqrt{L})f\|_p \leq c2^{-js}\|f\|_{B^s_{p,q}}$, for all $j \geq 0$, and so $\sum_{j=0}^{\infty} \varphi_j(\sqrt{L})f \in L^p$, with $\|\sum_{j=0}^{\infty} \varphi_j(\sqrt{L})f\|_p \leq c\|f\|_{B^s_{p,q}}$. At the same time, with the special choice of the pair of admissible cut-off functions, we also have by Theorem 5.13(ii) that $f = \sum_{j=0}^{\infty} \varphi_j(\sqrt{L})f$ in $\mathcal{D}'$. Therefore $f \in L^p$, and the Littlewood-Paley decomposition holds

$$f \underset{L^p}{=} \sum_{j=0}^{\infty} \varphi_j(\sqrt{L})f.$$

Since $\sum_{j=0}^{m-1} \varphi_j(\sqrt{L})f \in \Sigma^p_{2^m}$, $m \geq 1$, we obtain an estimate for the linear approximation from $\Sigma^p_{2^m}$ defined in (3.24)

$$\mathcal{E}_{2^m}(f)_p \leq \sum_{j=m}^{\infty} \|\varphi_j(\sqrt{L})f\|_p. \tag{5.27}$$

Using (5.27) we may apply the discrete Hardy inequality (see e.g [**21**, Lemma 2.3.4]) to obtain for any $0 < q < \infty$

$$\sum_{m=0}^{\infty} (2^{ms}\mathcal{E}_{2^m}(f)_p)^q \leq c \sum_{j=0}^{\infty} (2^{js}\|\varphi_j(\sqrt{L})f\|_p)^q.$$

Using (5.27) also provides for $q = \infty$

$$\sup_{m \geq 0} 2^{ms}\mathcal{E}_{2^m}(f)_p \leq c \sup_{j \geq 0} 2^{js}\|\varphi_j(\sqrt{L})f\|_p.$$

Thus, we may conclude the first embedding $\|f\|_{A^s_{p,q}} \leq c\|f\|_{B^s_{p,q}}$.

In the other direction, let $f \in A^s_{p,q}$. Note that for any $g \in \Sigma^p_{2^{j-1}}$ and $j \geq 1$, $\varphi_j(\sqrt{L})g = 0$. Therefore, for any $f \in L^p$, using also Proposition 1.12

$$\|\varphi_j(\sqrt{L})f\|_p = \|\varphi_j(\sqrt{L})(f-g)\|_p \leq c\|f-g\|_p.$$

This yields

$$\|\varphi_j(\sqrt{L})f\|_p \leq c\mathcal{E}_{2^{j-1}}(f)_p, \quad \forall j \geq 1.$$

Obviously, we also have by Proposition 1.12 the estimate $\|\varphi_0(\sqrt{L})f\|_p \leq c\|f\|_p$. Together, these estimates give $\|f\|_{B^s_{p,q}} \leq c\|f\|_{A^s_{p,q}}$. □

5.5. Lipschitz-Besov Spaces

Let $\varphi_0, \varphi \in C_0^\infty(\mathbb{R})$ be any admissible pair in the sense of Definition 5.15 with $b > 1$. The spaces $B^s_{\infty,\infty}$, $s > 0$, with the norm

$$\|f\|_{B^s_{\infty,\infty}} = \sup_{j \geq 0} 2^{js} \|\varphi_j(\sqrt{L}) f\|_\infty,$$

can be identified as **Lipschitz spaces**. Indeed, we provided in §1.1.2, the example of the case $M = \mathbb{R}^d$, where this Littlewood-Paley characterization is equivalent to the classical definition of Lipschitz spaces through finite differences.

Let us see this in our general setup. The classical Lipschitz spaces, $\mathrm{Lip}(s)$, $0 < s < 1$, are defined as the set of all $f \in L^\infty(M)$, such that

$$\|f\|_{\mathrm{Lip}(s)} := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^s} < \infty.$$

Here, we extend the range to $s > 0$ (putting aside the notion of higher order differences or derivatives as in [**32**, §1.4]). For the characterization, we first recall the following. By Proposition 3.3, for any $g \in \Sigma_\lambda^\infty$, $\lambda \geq 1$

$$|g(x) - g(y)| \leq c\|g\|_\infty (\lambda\rho(x,y))^\alpha, \quad \forall x, y \in M. \tag{5.28}$$

Theorem 5.23. *Assume the Markov property* (2.42). *Then, the following continuous embeddings hold: For any* $s > 0$

$$Lip(s) \subseteq B^s_{\infty,\infty},$$

and for any $0 < s < \alpha$*, where* α *is the structural constant from* (2.2)

$$B^s_{\infty,\infty} \subseteq Lip(s).$$

PROOF. Let $f \in Lip(s)$ and choose $\varphi \in C_0^\infty(\mathbb{R})$, even, with $\varphi \equiv 1$ on $[-1,1]$, $0 \le \varphi \le 1$ and $\operatorname{supp}(\varphi) \subset [-2,2]$. Since $f \in L^\infty$, by Proposition 1.12, $\varphi(\lambda^{-1}\sqrt{L})f \in L^\infty$, for all $\lambda \ge 1$. Using (2.46), (2.28) with $\lambda \ge 1$, $k > s + 3d/2$ and (1.33), we derive for any $x \in M$

$$\begin{aligned} |\varphi(\lambda^{-1}\sqrt{L})f(x) - f(x)| &= \left| \int_M \varphi(\lambda^{-1}\sqrt{L})(x,y)(f(y) - f(x))d\mu(y) \right| \\ &\le c\|f\|_{Lip(s)} \int_M D_{\lambda^{-1},k}(x,y)\rho(x,y)^s d\mu(y) \\ &\le c\lambda^{-s}\|f\|_{Lip(s)} \int_M D_{\lambda^{-1},k-s}(x,y)d\mu(y) \\ &\le c\lambda^{-s}\|f\|_{Lip(s)}. \end{aligned}$$

Since $\varphi(\lambda^{-1}\sqrt{L})f \in \Sigma_{2\lambda}^{\infty}$, $\forall \lambda \geq 1$, this implies f is in the approximation space associated with linear approximation from spaces of finite spectra $A_{\infty,\infty}^{s}$ (see Definition 5.21), with $\|f\|_{A_{\infty,\infty}^{s}} \leq c\|f\|_{Lip(s)}$. By Theorem 5.22 this means $f \in B_{\infty,\infty}^{s}$, with $\|f\|_{B_{\infty,\infty}^{s}} \leq c\|f\|_{Lip(s)}$.

We turn to prove the inverse embedding for $0 < s < \alpha$. We use a construction of cut-off functions we have used before, where $\varphi_0 \in C_0^{\infty}(\mathbb{R})$, even, with $\varphi_0 \equiv 1$ on $[-1,1]$, $0 \leq \varphi_0 \leq 1$ and $\operatorname{supp}(\varphi_0) \subset [-2,2]$. Then we construct $\varphi := \varphi_0 - \varphi_0(2\cdot)$ and $\varphi_j := \varphi(2^{-j}\cdot)$, for $j \geq 1$. This give the partition of unity $\sum_{j=0}^{\infty} \varphi_j \equiv 1$. For $f \in B_{\infty,\infty}^{s}$, we have $\varphi_0(\sqrt{L})f \in \Sigma_{2}^{\infty}$ and $\varphi_j(\sqrt{L})f \in \Sigma_{2^{j+1}}^{\infty}$, for $j \geq 1$. We also have

$\|\varphi_j(\sqrt{L})f\|_\infty \le c2^{-js}\|f\|_{B^s_{\infty,\infty}}$, for all $j \ge 0$. This means that the Littlewood-Paley representation of f converges in L^∞ to f

$$f(x) = \sum_{j=0}^{\infty} \varphi_j(\sqrt{L})f(x), \quad \forall x \in M,$$

and that

$$\|f\|_\infty \le c\|f\|_{B^s_{\infty,\infty}}. \tag{5.29}$$

Now, assume $x, y \in M$, with $0 < \rho(x,y) \le 1$. Let $J \ge 0$, such that $2^{-(J+1)} \le \rho(x,y) \le 2^{-J}$. Applying (5.28) gives

$$|\varphi_j(\sqrt{L})f(x) - \varphi_j(\sqrt{L})f(y)| \le c\|\varphi_j(\sqrt{L})f\|_\infty (2^j \rho(x,y))^\alpha, \quad 0 \le j \le J-1.$$

This leads to

$$\begin{aligned}
|f(x)-f(y)| &\le \sum_{j=0}^{\infty} |\varphi_j(\sqrt{L})f(x)-\varphi_j(\sqrt{L})f(y)| \\
&\le c\sum_{j=0}^{J-1} \|\varphi_j(\sqrt{L})f\|_\infty (2^j\rho(x,y))^\alpha + 2\sum_{j=J}^{\infty} \|\varphi_j(\sqrt{L})f\|_\infty \\
&\le c\|f\|_{B^s_{\infty,\infty}} \left(\sum_{j=0}^{J-1} 2^{-js}(2^j\rho(x,y))^\alpha + \sum_{j=J}^{\infty} 2^{-js}\right) \\
&\le c\|f\|_{B^s_{\infty,\infty}} \left(\rho(x,y)^\alpha 2^{(\alpha-s)J} + 2^{-Js}\right) \\
&\le c\|f\|_{B^s_{\infty,\infty}} \left(\rho(x,y)^\alpha \rho(x,y)^{s-\alpha} + \rho(x,y)^s\right) \\
&\le c\|f\|_{B^s_{\infty,\infty}} \rho(x,y)^s.
\end{aligned}$$

The case $\rho(x, y) \geq 1$ is simpler. Indeed, using (5.29)

$$\frac{|f(x) - f(y)|}{\rho(x, y)^s} \leq 2\|f\|_\infty \leq c\|f\|_{B^s_{\infty,\infty}}.$$

$\square$