Introduction to function spaces: Assignment I

1. [Distribution function] Let $f, g : X \to \mathbb{R}$ be measurable and $\alpha, \beta > 0$. Prove the following properties:

   (i) $d_{cf}(\alpha) = d_f(\alpha/|c|)$, $\forall c \neq 0$,

   (ii) $d_{f+g}(\alpha+\beta) \leq d_f(\alpha) + d_g(\beta)$.

2. [Schwartz class] Prove that $\varphi \in S$ iff $\varphi \in C^\infty(\mathbb{R}^n)$ and for any $\alpha \in \mathbb{Z}_+^n$, and $N > 0$, $\exists C_\varphi(\alpha, N) > 0$ such that
$$\left|\partial^\alpha \varphi(x)\right| \leq C_\varphi(\alpha, N)\left(1+|x|\right)^{-N}.$$

3. [Schwartz class] Prove that if $\varphi \in S$, then $x^\beta \varphi(x) \in S$, $\forall \beta \in \mathbb{Z}_+^n$.

4. [Distributions] Prove that $f(x) = \sin(4x)x^5 \in S'(\mathbb{R})$.

5. [Distributional derivative] Prove
$$H'(x) = \begin{cases} 1, & -1 \leq x < 0, \\ -1, & 0 \leq x \leq 1, \\ 0, & \text{else}, \end{cases} \qquad \text{where } H(x) := \begin{cases} x+1, & -1 \leq x < 0, \\ 1-x, & 0 \leq x \leq 1, \\ 0, & \text{else}. \end{cases}$$

6. Assume there exists a constant $C(r) > 0$, such that for any $g \in C^r(\mathbb{T})$
$$\|g - S_N g\|_{L_2(\mathbb{T})} \leq C(r) N^{-r} |g|_{r,2}.$$
   Prove using density of $C^r(\mathbb{T})$, that this estimate holds for any $f \in W_2^r(\mathbb{T})$. Use a sequence $\{f_k\}_{k\geq 1}$, $f_k \in C^r(\mathbb{T})$, $\|f - f_k\|_{W_2^r(\mathbb{T})} \underset{k\to\infty}{\to} 0$.