Foundations of Approximation Theory Spring 2017:

Theorem list for the exam

Theorems for 20 points

1. [Piecewise constant approximation] Prove that for $g \in W_p^1(\mathbb{R})$, $1 \le p \le \infty$,

$$E\left(g, S(N_1)^h\right)_p \le h\|g'\|_p , \quad h > 0 .$$

2. [Discrete Besov norm] Define the integral form of the Besov semi-norm (over $[0,1]$) . Prove the equivalency with the discrete dyadic form.

3. [Refinability of B-splines] Show that for $r \ge 1$, the univariate B-spline N_r, satisfies the two-scale relation

$$N_r(x) = \sum_{k=0}^{r} 2^{1-r} \binom{r}{k} N_r(2x-k), \qquad \forall x \in \mathbb{R} .$$

4. [Band-limited functions] The Sinc function is defined by $\phi(x) := \mathcal{F}^{-1}\left(\mathbf{1}_{[-\pi,\pi]^n}(\cdot)\right)(x)$.

   (i) Prove that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^n}$ is an ortho-basis of $S(\phi)$.

   (ii) Prove that if $f \in L_2(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, and $f \in S(\phi)$, then $f(x) = \sum_{k\in\mathbb{Z}^n} f(k)\phi(x-k)$.

5. [Nikolskii-type equivalence over convex domains] Prove that for any $n, r \ge 1$ and $0 < p, q \le \infty$, there exist constants of equivalence that depend only on these parameters, such that for any bounded convex domain $\Omega \subset \mathbb{R}^n$ and any algebraic polynomial $P \in \Pi_{r-1}(\mathbb{R}^n)$

$$\|P\|_{L_q(\Omega)} \sim |\Omega|^{1/q-1/p} \|P\|_{L_p(\Omega)} .$$

Comment: You may use John's Theorem and the equivalence of finite dimensional Banach spaces.

Let $\{\phi_m\}$ be a sequence of functions in $L_p(\mathbb{R}^n)$, $0 < p < \infty$, that satisfy

(i) $\operatorname{supp}(\phi_m) \subseteq E_m$, with $0 < |E_m| < \infty$ and $\|\phi_m\|_\infty \le c|E_m|^{-1/p} \|\phi_m\|_p$,

(ii) If $x \in E_m$, then

$$\sum_{x\in E_j, |E_j| \ge |E_m|} \left(\frac{|E_m|}{|E_j|}\right)^{1/p} \le c \ .$$

6. Under the above assumptions, let $F = \sum_{j\in\Lambda} \phi_j$, with $\# \Lambda \le N$ and $\|\phi_j\|_p \le L$, for $j \in \Lambda$. Prove that

$$\|F\|_p \le cL(\#\Lambda)^{1/p}.$$

7. Under the above assumptions, denote (formally) $f = \sum_m \phi_m$ and assume that for $0 < \tau < p$,

   $N(f) := \left(\sum_m \|\phi_m\|_p^\tau\right)^{1/\tau} < \infty$. Then, prove that with the rearrangement $\|\phi_{m_1}\|_p \ge \|\phi_{m_2}\|_p \ge \cdots$, we have that

$$\|f - S_M\|_p \le cM^{-\alpha} N(f), \qquad \text{for } S_M := \sum_{j=1}^{M} \phi_{m_j}.$$

8. [Jackson theorem for Wavelets] Let $f \in L_2(\mathbb{R}) \cap B_\tau^\alpha(\mathbb{R})$, $1/\tau = \alpha + 1/2$. Let $\{\psi_I\}, \{\tilde{\psi}_I\}$ be a compactly supported, piecewise polynomial, biorthogonal wavelet and dual wavelet basis. Let $\sigma_N(f)_p := \inf_{g \in \Sigma_M} \|f - g\|_2$, where Σ_M is the collection of M - term (or less) wavelets. Prove that

$$\sigma_M(f)_2 \le cM^{-\alpha} |f|_{B_\tau^\alpha}.$$

Comment: You may use the results of (7), but you need to explain why the conditions hold. You may use the equivalence

$$|f|_{B_\tau^\alpha} \sim \left(\sum_I |\langle f, \tilde{\psi}_I \rangle|^\tau\right)^{1/\tau}.$$

9. [Bernstein inequality for Wavelets] Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, $f = \sum_{I \in \Lambda} c_{I,p} \psi_{I,p}$ with $\#\Lambda \le N$. Then show

$$|f|_{B_\tau^\alpha(L_\tau(\mathbb{R}))} \le cN^\alpha \|f\|_p, \qquad \frac{1}{\tau} = \alpha + \frac{1}{p}.$$

Comment: You may use

$$\|f\|_p \sim \|S(f,\cdot)\|_p, \qquad S(f,x) := \left(\sum_I c_{I,p}(f)^2 |I|^{-2/p} \chi_I(x)\right)^{1/2}.$$

Theorems for 30 points

10. [Bernstein for trigonometric polynomials]. Prove that for any univariate real trigonometric polynomial P of degree N:
    (i) $P'(t)^2 + N^2 P(t)^2 \le N^2 \|P\|_\infty^2$, $t \in [-\pi, \pi]$.
    (ii) $\|P^{(k)}\|_\infty \le N^k \|P\|_\infty$, $k \ge 1$.

11. [Trigonometric approximation of piecewise algebraic polynomials] Let $f \in L_\infty[-\pi,\pi]$, $\|f\|_\infty \le 1$, be a piecewise algebraic polynomial $f(t) = \sum_{k=1}^{M} P_k(t)\mathbf{1}_{(t_k,t_{k+1}]}(t)$, where $-\pi < t_1 < t_2 < \cdots < t_{M+1} < \pi$, are knots and P_k , $k = 1,\dots,M$, are algebraic polynomials of degree $r-1$, i.e. $P_k(t) = \sum_{j=0}^{r-1} a_{k,j} t^j$.

(i) Prove that for sufficiently large N , $E_N(f)_p \le C(r,p)\left(\frac{M}{N}\right)^{1/p}$, $1 \le p < \infty$, where $E_N(f)_p$ is the degree of approximation using trigonometric polynomials of degree N . You may assume the Jackson inequality.

(ii) Prove that $f \in B_\tau^\alpha(L_\tau)$, for any $\alpha > 0$, with $1/\tau = \alpha + 1/p$.

12. [Equivalence of modulus of smoothness K-functional] Let $1 \le p \le \infty$.

(i) Prove that for any $g \in W_p^r(\mathbb{R})$, we have $\omega_r(g,t)_p \le ct^r |g|_{r,p}$, $t > 0$.

(ii) Prove that for any $f \in L_p(\mathbb{R})$, we have $\omega_r(f,t)_p \le cK_r(f,t^r)_p$, $t > 0$.

Comment: You may use the Minkowski integral inequality.

13. [Kernel approximation] Assume a kernel operator T , with kernel $k(x,y)$ satisfies for $r \ge 1$

(i) $P(x) = TP(x) = \int_{\mathbb{R}^n} k(x,y)P(y)dy$, $\forall P \in \Pi_{r-1}(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$.

(ii) $|k(x,y)| \le c\frac{1}{(1+|x-y|)^{n+r+\varepsilon}}$, for some $\varepsilon > 0$ and any $x, y \in \mathbb{R}^n$.

Prove that for $f \in W_\infty^r(\mathbb{R}^n)$ and $T_h f(x) := \int_{\mathbb{R}^n} h^{-n} k(h^{-1}x, h^{-1}y) f(y) dy$,

$$\|f - T_h f\|_\infty \le ch^r |f|_{r,\infty}, \qquad h > 0,$$

Comment: You may use the Taylor remainder estimate for $p = \infty$

$$R_{r,x} f(y) \le c|y-x|^r \max_{z \in B(x,|y-x|)} \max_{|\alpha|=r} |\partial^\alpha f(z)|.$$

14. [Spectral L_2 approximation of the Sinc] The Sinc function is defined by $\phi(x) := \mathcal{F}^{-1}\left(\mathbf{1}_{[-\pi,\pi]^n}(\cdot)\right)(x)$. Then show that for any $r \ge 1$, $h > 0$ and $f \in W_2^r(\mathbb{R}^n)$

$$E\left(f, S(\phi)^h\right)_2 \le ch^r |f|_{r,2}.$$

15. [Jackson theorem for trigonometric polynomials]. Prove that for a periodic function $f \in L_p([-\pi,\pi])$, $1 \le p \le \infty$, and any $r \ge 1$

$$E_N(f)_p \le C(r)\omega_r(f, N^{-1})_p,$$

where $E_N(f)_p$ is the degree of approximation by trigonometric polynomials of degree N.

Comment: For the Jackson kernel $J_{N,r}$, you may assume the estimate

$$\int_0^\pi t^k J_{N,r}(t)\,dt \le C(r) N^{-k}, \; k = 0,\dots,2r-2.$$

16. [Jackson & Bernstein machinery] Let the sequence $\Phi := \{\Phi_N\}_{N\ge 0} \subset X$, where X is a Banach space, satisfy

(i) $0 \in \Phi_N, \Phi_0 = 0$,

(ii) $\Phi_N \subset \Phi_{N+1}$,

(iii) $a\Phi_N = \Phi_N$, $\forall a \ne 0$.

(iv) $\Phi_N + \Phi_N \subset \Phi_{cN}$, for some fixed $c > 0$,

(v) $\bigcup_N \Phi_N$ is dense in X,

We denote $E_N(f)_X := \min_{\varphi\in\Phi_N} \|f-\varphi\|_X$. For $r \ge 1$, let $Y = Y_r \subset X$ and assume that the Jackson and Bernstein inequalities hold:

(i) $E_N(g)_X \le cN^{-r}|g|_Y$, $\forall g \in Y$,

(ii) $|\varphi|_Y \le cN^r\|\varphi\|_X$, $\forall \varphi \in \Phi_N$.

Then prove the characterization of the approximation space for any $0 < \alpha < r$, $1 \le q < \infty$,

$$A_q^\alpha(X) = (X,Y)_{\alpha/r,q}.$$

Comments

(i) You may use the discrete form of the semi-norms

$$|f|_{A_q^\alpha} \sim \left(\sum_{m=0}^\infty \left(2^{m\alpha} E_{2^m}(f)\right)^q\right)^{1/q}, \quad |f|_{\theta,q} \sim \left(\sum_{m=0}^\infty \left(2^{m\theta r} K(f, 2^{-mr})\right)^q\right)^{1/q}.$$

(ii) You may use the discrete Hardy inequality: For non-negative sequences $\{a_m\}_{m=0}^\infty, \{b_m\}_{m=0}^\infty$, if $a_m \le c2^{-mr}\sum_{k=0}^m 2^{kr} b_k$, $\forall m \ge 0$, then $\left(\sum_{m=0}^\infty \left(2^{m\alpha} a_m\right)^q\right)^{1/q} \le c\left(\sum_{m=0}^\infty \left(2^{m\alpha} b_m\right)^q\right)^{1/q}$.