1.5. Non Negative Self Adjoint Operators

In this section we review some aspects of functional analysis theory which serve as the foundation to the generalized approach we take [**55, 60**]. The theory is presented for a general Hilbert space $\mathcal{H}$, but for most part, will later be applied in the case $\mathcal{H} = L^2(M)$.

Definition 1.16. We assume L is a linear, possibly unbounded operator acting on its domain $D(L)$ in a Hilbert space $\mathcal{H}$. We assume

(i) $D(L)$ is dense in $\mathcal{H}$,
(ii) $Lf \in \mathcal{H}$, for any $f \in D(L)$.

60. K. Yosida, *Functional Analysis*, Springer, Berlin, 1980.

We shall say L is a **Self-Adjoint** operator on $D(L) \subset \mathcal{H}$ if for any $f, g \in D(L)$ we have $\langle Lf, g\rangle = \langle f, Lg\rangle$. We shall say L is **Non Negative** if for any $f \in D(L)$, we have $\langle Lf, f\rangle \geq 0$. It is common to use the notation $L \geq 0$ to denote a non negative operator L. We shall say L is **Positive** if for any $f \in D(L)$, $f \neq 0$, we have $\langle Lf, f\rangle > 0$. In some cases the notation $L > 0$ is used to denote a positive operator L.

1.5.1. Resolution of Identity.

To simplify, we use the notation of $\lim_{\lambda'\to\lambda} T_{\lambda'}$ for the limit of a sequence of operators $T_{\lambda'}$ in the strong operator topology.

DEFINITION 1.22. A family of orthogonal projections E_λ, $-\infty < \lambda < \infty$, in a Hilbert space $\mathcal{H}$, is called a real **Resolution of Identity** if it satisfies the conditions:

$$E_\lambda E_{\lambda'} = E_{\min(\lambda,\lambda')},$$

$$E_{-\infty} := \lim_{\lambda\to-\infty} E_\lambda = 0, \quad E_\infty := \lim_{\lambda\to\infty} E_\lambda = I,$$

$$\lim_{\lambda'\to\lambda} E_{\lambda'} = E_\lambda.$$

It is readily seen that in a separable Hilbert space $\mathcal{H}$, any countable orthonormal basis in any arbitrary ordering $\{f_j\}_{j=1}^{\infty}$, yields a resolution of identity by

$$E_\lambda f := \begin{cases} 0, & \lambda \le 0, \\ \sum_{j\le\lambda} \langle f, f_j\rangle_{\mathcal{H}} f_j, & \lambda > 0. \end{cases}$$

EXAMPLE 1.23. For $M = \mathbb{R}^d$, let $\{E_\lambda\}$ be defined for any $f \in L^2(\mathbb{R}^d)$ by

$$E_\lambda f(x) := \begin{cases} 0, & \lambda \le 0, \\ \frac{1}{(2\pi)^d} \int_{[-\pi\lambda, \pi\lambda]^d} \hat{f}(\omega) e^{i\omega x} d\omega, & \lambda > 0. \end{cases}$$

Then $\{E_\lambda\}$ is a resolution of identity.

PROOF. Recall the sinc function

$$\phi(x) := \prod_{j=1}^{d} \frac{\sin(\pi x_j)}{\pi x_j}.$$

It is easy to see that with $\phi_\lambda := \lambda^d \phi(\lambda \cdot)$, $\lambda > 0$, we have

$$\widehat{\phi_\lambda} = \mathbf{1}_{[-\pi\lambda, \pi\lambda]^d}.$$

We claim that for $\lambda > 0$, $E_\lambda f = f * \phi_\lambda$, for all $f \in L^2(\mathbb{R}^d)$. Indeed, it is easy to see from the definition that $E_\lambda f$ is the inverse Fourier transform of $\hat{f}\widehat{\phi_\lambda}$ and so

$$\widehat{E_\lambda f} = \hat{f}\widehat{\phi_\lambda} = \widehat{f * \phi_\lambda}.$$

Therefore, for $\lambda, \lambda' > 0$

$$\begin{aligned}\widehat{E_\lambda E_{\lambda'} f} &= f * \widehat{\phi_\lambda * \phi_{\lambda'}} \\ &= \hat{f}\widehat{\phi_\lambda}\widehat{\phi_{\lambda'}} \\ &= \hat{f}\mathbf{1}_{[-\pi\lambda,\pi\lambda]^d}\mathbf{1}_{[-\pi\lambda',\pi\lambda']^d} \\ &= \hat{f}\mathbf{1}_{[-\pi\min(\lambda,\lambda'),\pi\min(\lambda,\lambda')]^d} \\ &= \widehat{E_{\min(\lambda,\lambda')} f}.\end{aligned}$$

□

For ease of notation we define for $-\infty < \lambda < \lambda' < \infty$, the orthogonal projection

$$E(\lambda, \lambda'] := E_{\lambda'} - E_{\lambda}.$$

PROPOSITION 1.24. For any fixed $f, g \in \mathcal{H}$, the function $h(\lambda) := \langle E_\lambda f, g\rangle$ is a function of bounded variation.

PROOF. Take any partition $\lambda_1 < \cdots < \lambda_N$. We use the fact that for any $-\infty < \lambda < \lambda' < \infty$, $E(\lambda, \lambda']$ is an orthogonal projection and then apply the continuous and discrete Schwartz inequalities to obtain

$$\begin{aligned}
\sum_{n=1}^{N-1} |\langle E(\lambda_n, \lambda_{n+1}]f, g\rangle| &= \sum_{n=1}^{N-1} |\langle E(\lambda_n, \lambda_{n+1}]f, E(\lambda_n, \lambda_{n+1}]g\rangle| \\
&\leq \sum_{n=1}^{N-1} \|E(\lambda_n, \lambda_{n+1}]f\|_{\mathcal{H}} \|E(\lambda_n, \lambda_{n+1}]g\|_{\mathcal{H}} \\
&\leq \left(\sum_{n=1}^{N-1} \|E(\lambda_n, \lambda_{n+1}]f\|_{\mathcal{H}}^2\right)^{1/2} \left(\sum_{n=1}^{N-1} \|E(\lambda_n, \lambda_{n+1}]g\|_{\mathcal{H}}^2\right)^{1/2} \\
&= \left(\|E(\lambda_1, \lambda_N]f\|_{\mathcal{H}}^2 \|E(\lambda_1, \lambda_N]g\|_{\mathcal{H}}^2\right)^{1/2} \\
&\leq \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}.
\end{aligned}$$

Proposition 1.25. Let $\varphi : [\alpha, \beta] \to \mathbb{C}$ be continuous and let $f \in \mathcal{H}$. Then, we can define

$$\int_{\alpha}^{\beta} \varphi(\lambda) dE_{\lambda} f,$$

as the limit of the Riemann sums

$$\sum_{n=1}^{N-1} \varphi(\lambda'_n) E(\lambda_n, \lambda_{n+1}] f, \quad \alpha = \lambda_1 < \cdots \lambda_N = \beta, \quad \lambda'_n \in (\lambda_n, \lambda_{n+1}],$$

where $\max_n(\lambda_{n+1} - \lambda_n) \to 0$ as $N \to \infty$.

PROOF. We will show that the limit exists and is unique. Since φ is uniformly continuous on $[\alpha, \beta]$, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|\varphi(\lambda) - \varphi(\lambda')| < \varepsilon$, whenever $\alpha \leq \lambda, \lambda' \leq \beta$, $|\lambda - \lambda'| < \delta$. Let

$$\alpha = \lambda_1 < \cdots < \lambda_N = \beta, \quad \alpha = \gamma_1 < \cdots < \gamma_K = \beta,$$

be any two partitions such that

$$\lambda_{n+1} - \lambda_n < \delta, \quad 1 \leq n \leq N-1, \quad \gamma_{k+1} - \lambda_k < \delta, \quad 1 \leq k \leq K-1,$$

and let

$$\alpha = \sigma_1 < \cdots < \sigma_J = \beta, \quad J \leq M + K,$$

be the superposition of these two partitions. If $\lambda_n' \in (\lambda_n, \lambda_{n+1}]$ and $\gamma_k' \in (\gamma_k, \gamma_{k+1}]$, then

$$\begin{aligned} \left\| \sum_{n=1}^{N-1} \varphi(\lambda_n') E(\lambda_n, \lambda_{n+1}] f - \sum_{k=1}^{K-1} \varphi(\gamma_k') E(\gamma_k, \gamma_{k+1}] f \right\|_{\mathcal{H}}^2 &\leq \varepsilon^2 \left\| \sum_{j=1}^{J-1} E(\sigma_j, \sigma_{j+1}] f \right\|_{\mathcal{H}}^2 \\ &= \varepsilon^2 \|E(\alpha, \beta] f\|_{\mathcal{H}}^2 \\ &\leq \varepsilon^2 \|f\|_{\mathcal{H}}^2 \end{aligned}$$

Corollary 1.26. For a resolution of identity $\{E_\lambda\}$, a function $\varphi : \mathbb{R} \to \mathbb{C}$, that is piecewise continuous on intervals $\{(\gamma_k, \gamma_{k+1}]\}_{k=-\infty}^{\infty}$, and any $f \in \mathcal{H}$, we may define for any $-\infty < \alpha < \beta < \infty$,

$$\int_\alpha^\beta \varphi(\lambda) dE_\lambda f := \sum_k \int_{(\gamma_k, \gamma_{k+1}] \cap [\alpha, \beta]} \varphi(\lambda) dE_\lambda f$$

and then

$$\int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda f := \lim_{\alpha \to -\infty, \beta \to \infty} \int_\alpha^\beta \varphi(\lambda) dE_\lambda f, \tag{1.43}$$

if the limit exists.

We also have

Corollary 1.27. For any resolution of identity $\{E_\lambda\}$, continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ and any Borel set $U \subset \mathbb{R}$, the operator

$$\int_U \varphi(\lambda) dE_\lambda,$$

is well defined as the limit of the Riemann sums over intervals.

THEOREM 1.28. *For any resolution of identity* $\{E_\lambda\}$, *a given* $f \in \mathcal{H}$ *and a piecewise continuous function* $\varphi : \mathbb{R} \to \mathbb{C}$, *the following conditions are equivalent*

(i) *The following integral exists as a limit in the sense of* (1.43)

$$\int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda f,$$

(ii) *The following holds*

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\|E_\lambda f\|_{\mathcal{H}}^2 < \infty,$$

(iii) *The following defines a bounded linear functional*

$$F(g) := \int_{-\infty}^{\infty} \varphi(\lambda) d\langle E_\lambda f, g\rangle, \quad \forall g \in \mathcal{H}.$$

THEOREM 1.29. *Let* $\{E_\lambda\}$ *be a resolution of identity and* $\varphi : \mathbb{R} \to \mathbb{C}$ *be piecewise continuous. Let*

$$D := \{f \in \mathcal{H} : \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\|E_\lambda f\|_{\mathcal{H}}^2 < \infty\}.$$

Then, the adjoint of the operator $L := \int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda$*, defined by*

$$\langle Lf, g\rangle = \int_{-\infty}^{\infty} \varphi(\lambda) d\langle E_\lambda f, g\rangle, \quad \forall f \in D, g \in \mathcal{H}, \tag{1.44}$$

is given by

$$\langle L^* f, g\rangle = \int_{-\infty}^{\infty} \overline{\varphi(\lambda)} d\langle E_\lambda f, g\rangle, \quad \forall f \in D(L^*), g \in \mathcal{H}. \tag{1.45}$$

Therefore, if φ *is real, then* L *is self-adjoint.*

PROOF. Let $f \in D(L^*)$, with $L^* f = f^*$. It is readily seen that for any $g \in \mathcal{H}$, $E_{[\alpha,\beta]} g \in D$, for any $-\infty < \alpha < \beta < \infty$. Therefore,

$$\begin{aligned}
\langle E_{[\alpha,\beta]} g, f^* \rangle &= \langle E_{[\alpha,\beta]} g, L^* f \rangle \\
&= \langle L E_{[\alpha,\beta]} g, f \rangle \\
&= \int_{\alpha}^{\beta} \varphi(\lambda) d\langle E_\lambda g, f \rangle.
\end{aligned}$$

Taking the limit, $\alpha \to -\infty$, $\beta \to \infty$ implies that

$$\int_{-\infty}^{\infty} \varphi(\lambda) d\langle E_\lambda g, f \rangle = \langle g, f^* \rangle.$$

Thus we may define a bounded linear functional by

$$\begin{aligned}
F(g) :=& \overline{\langle g, f^* \rangle} \\
&= \int_0^\infty \overline{\varphi(\lambda)} d\overline{\langle E_\lambda g, f \rangle} \\
&= \int_0^\infty \overline{\varphi(\lambda)} \langle E_\lambda f, g \rangle.
\end{aligned}$$

By Theorem 1.28, this implies $f \in D$, and that $\int_{-\infty}^{\infty} \overline{\varphi(\lambda)} dE_\lambda f$ exists as a limit in the sense of (1.43). Therefore, for any $g \in \mathcal{H}$

$$\begin{aligned} \langle L^* f, g \rangle &= \overline{\langle g, f^* \rangle} \\ &= \int_{-\infty}^{\infty} \overline{\varphi(\lambda)} \langle E_\lambda f, g \rangle. \end{aligned}$$

□

Corollary 1.30. Let $\varphi : \mathbb{R} \to \mathbb{C}$, be piecewise continuous. For $L = \int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda$ given by (1.44), we have for any $f \in D$

$$\|Lf\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\|E_\lambda f\|_{\mathcal{H}}^2, \tag{1.46}$$

In particular, for $I = \int_{-\infty}^{\infty} dE_\lambda$

$$\|f\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} d\|E_\lambda f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}, \tag{1.47}$$

and for $\varphi \in L^\infty(\mathbb{R})$

$$\|L\|_{\mathcal{H}\to\mathcal{H}} \leq \|\varphi\|_\infty. \tag{1.48}$$

Proof. Using (1.45)

$$\begin{aligned}
\|Lf\|_{\mathcal{H}}^2 &= \int_{-\infty}^{\infty} \varphi(\lambda) d\langle E_\lambda f, Lf\rangle \\
&= \int_{-\infty}^{\infty} \varphi(\lambda) d\langle L^* E_\lambda f, f\rangle \\
&= \int_{-\infty}^{\infty} \varphi(\lambda) d_\lambda \left(\int_{-\infty}^{\infty} \overline{\varphi(u)} d_u \langle E_u E_\lambda f, f\rangle \right) \\
&= \int_{-\infty}^{\infty} \varphi(\lambda) d_\lambda \left(\int_{-\infty}^{\lambda} \overline{\varphi(u)} d_u \langle E_u f, f\rangle \right) \\
&= \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\langle E_\lambda f, f\rangle \\
&= \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\|E_\lambda f\|_{\mathcal{H}}^2.
\end{aligned}$$

1.5.2. Spectral Theorem for Self-Adjoint Operators.

Definition 1.32. In the special case $\varphi(\lambda) = \lambda$, we have a self-adjoint operator L defined through

$$\langle Lf, g\rangle = \int_{-\infty}^{\infty} \lambda d\langle E_\lambda f, g\rangle, \quad \forall f \in D(L), g \in \mathcal{H}.$$

We shall write it symbolically

$$L = \int_{-\infty}^{\infty} \lambda dE_\lambda,$$

and call it the **spectral resolution** or the **spectral representation** of the self-adjoint operator L.

One of the main results in functional analysis is that the inverse is also true (see [**61**, Theorem XI.6.1] for the proof)

THEOREM 1.33. *Let L be a self adjoint operator, where $L : D(L) \to \mathcal{H}$, with $D(L)$ a dense subset of a Hilbert space $\mathcal{H}$. Then, L has a uniquely determined spectral resolution*

$$L = \int_{-\infty}^{\infty} \lambda dE_\lambda = \int_{\sigma(L)} \lambda dE_\lambda, \tag{1.50}$$

where $\sigma(L) \subseteq \mathbb{R}$ is the spectrum of L (see Definition 1.17).

Recall that a non-negative self adjoint operator L on a Hilbert space $\mathcal{H}$ satisfies $\langle Lf, f\rangle \geq 0$, $\forall f \in D(L)$. Combining Theorem 1.33 and Proposition 1.21 give

COROLLARY 1.34. The spectral resolution of a non-negative self adjoint operator has the form

$$L = \int_{0}^{\infty} \lambda dE_\lambda. \tag{1.51}$$

We circle back to functional calculus, now constructed from the spectral resolution of a self-adjoint operator.

DEFINITION 1.35. For a self-adjoint operator L that admits a spectral resolution (1.50) and a piecewise continuous $\varphi : \mathbb{R} \to \mathbb{C}$, we will denote

$$\varphi(L) := \int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda. \tag{1.52}$$

PROPOSITION 1.36. let L be a self-adjoint operator that admits a spectral resolution (1.50) and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function, Then $\varphi(L)$ is also a self-adjoint operator.

PROOF. Essentially, this is derived from Theorem 1.29 whenever φ is real. Assume $f, g \in D(\varphi(L))$. Then,

$$\begin{aligned}\langle \varphi(L) f, g \rangle &= \int_{-\infty}^{\infty} \varphi(\lambda) d\langle E_\lambda f, g \rangle \\ &= \int_{-\infty}^{\infty} \varphi(\lambda) d\langle f, E_\lambda g \rangle \\ &= \langle f, \varphi(L) g \rangle.\end{aligned}$$

PROPOSITION 1.37. Let L be a self-adjoint operator that admits a spectral resolution (1.50), and let $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{C}$, be both piecewise continuous. Then

$$\varphi_1(L)\varphi_2(L) = \varphi_2(L)\varphi_1(L) = (\varphi_1\varphi_2)(L) = \int_{-\infty}^{\infty} \varphi_1(\lambda)\varphi_2(\lambda)dE_\lambda, \tag{1.53}$$

if the limit exists in the sense of (1.43).

DEFINITION 1.40. For a non-negative self-adjoint operator L, we call the operators

$$P_t := e^{-tL} = \int_0^\infty e^{-t\lambda} dE_\lambda, \quad t > 0,$$

the **Heat Kernel** operators.

The semi-group of heat kernel operators plays a crucial part in the dynamics of physics over manifolds. Consider the parabolic Cauchy problem associated with L on M, with some initial condition $u_0 \in L^2(M)$

$$\frac{\partial u}{\partial t} + Lu = 0, \quad t \geq 0,$$

$$u(x,0) = u_0(x), \quad \forall x \in M.$$

Then, using functional calculus over the spectral representation of L, it is easy to see that a weak-type solution is given by

$$u(\cdot,t) := e^{-tL}u_0 = \int_0^\infty e^{-t\lambda}dE_\lambda u_0, \quad t \ge 0.$$

Indeed,

$$\frac{\partial u}{\partial t} = -\int_0^\infty \lambda e^{-t\lambda}dE_\lambda u_0,$$

while by (1.53)

$$Lu(\cdot,t) = [Le^{-tL}]u_0 = \int_0^\infty \lambda e^{-t\lambda}dE_\lambda u_0.$$

As we shall see, the heat kernels also play a crucial part in our analysis. In fact, the centerpiece assumption of our setup is that these operators are indeed kernel operators, and that their kernels $\{p_t\}_{t>0}$ are continuous with excellent localization properties (see Definition 2.1). For now, we observe that by applying (1.48) with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(\lambda) = e^{-t\lambda}$, we see these operators are **contractive**, that is

$$\|P_t\|_{\mathcal{H}\to\mathcal{H}} \le 1, \quad \forall t > 0.$$